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NONLINEAR EQUATIONS OF MOTION FOR A PANEL SUBJECT TO EXTERNAL LOADS

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FOREWORD

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The equations of motion of a constant thickness, rectangular panel subject to external static loads and unsteady supersonic aerodynamic forces is derived. Both isotropic and orthotropic panels are examined. The static loads come from in-plane forces, a temperature differential, and a pressure differential. The equations allow for slight initial curvature due to manufacturing imperfections. The equations are derived using a nonlinear strain-displacement relationship which permits both linear and nonlinear panel response studies. First order piston theory predicts the unsteady aerodynamic forces due to the supersonic flow over the upper surface.

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SYMBOLS

English Nomenclature

a, b	plate streamwise and spanwise lengths (or length and width)
c, c_∞	speed of sound in disturbed and undisturbed flow
a_{ij}, b_{ij}, c_{ij}	generalized coordinates in x, y, and z directions
C_{VD}	viscous damping coefficient
D_x, D	orthotropic and isotropic plate bending stiffness
D_{xy}	orthotropic plate shear stiffness
E_x, E_y, E_z, E	orthotropic and isotropic Young's moduli
$E^* = E_y / E_x$	ratio of in-plane orthotropic moduli
G_{xy}, G_{yz}, G_{xz}, G	orthotropic and isotropic shear moduli
h	plate thickness
K_n, K_x, K_y, K_{xy}	curvature in n, x, y, and xy directions
$L = T - U$	Lagrangian
M	Mach number
$m_p = \rho_p h$	plate mass per unit area
m_n, m_x, m_y	slope in n, x, and y directions
N_x^a, N_y^a	applied in-plane loads in x and y directions
Q_{zij}	generalized nonconservative forces
T	kinetic energy
t	time
U	elastic energy
U_∞	freestream velocity
u, v	in-plane displacements in x and y directions
w	transverse displacement
x, y, z	spacial coordinates
$z_p(x, y)$	initial plate shape function

Greek Nomenclature

α_x, α_y, α_z, α	orthotropic and isotropic coefficients of thermal expansion
$\beta = (M^2 - 1)^{1/2}$	compressibility correction factor
α_i, β_j	in-plane modal functions in x and y directions
γ_{xy}, γ_{yz}, γ_{xz}	shear strains
ΔT, ΔP_S	static temperature and pressure differentials

$\epsilon_x, \epsilon_y, \epsilon_z$	normal strain in x, y, and z directions
λ	nondimensional dynamic pressure
μ	nondimensional mass ratio
$\nu_{xy}, \nu_{yz}, \nu_{xz}, \nu$	orthotropic and isotropic Poisson's ratios
ρ_∞, ρ_p	flow and plate mass densities
$\sigma_x, \sigma_y, \sigma_z$	normal stress in x, y, and z directions
τ	nondimensional time
$\tau_{xy}, \tau_{yz}, \tau_{xz}$	shear stresses
ϕ_i, ψ_j	transverse modal functions in x and y directions

SUMMARY

The equations of motion of a constant thickness, rectangular panel subject to external static loads and unsteady supersonic aerodynamic forces is derived. Both isotropic and orthotropic panels are examined. The static loads come from in-plane forces, a temperature differential, and a pressure differential. The equations allow for slight initial curvature due to manufacturing imperfections. The equations are derived using a nonlinear strain-displacement relationship which permits both linear and nonlinear panel response studies. First order piston theory predicts the unsteady aerodynamic forces due to the supersonic flow over the upper surface.

SECTION I

Introduction

The equations of motion for a panel are required to predict its response to various external loading conditions. The type of analysis to be performed is dependent upon the applied loading conditions. Some of the possible loads include in- and out-of-plane force distributions and temperature and pressure differentials. The equations developed here will allow for any combination of these loads.

The equations will be used in a panel flutter analysis in which the panel may be subject to all these loads simultaneously. The in-plane loads are assumed constant along a side. The temperature of both the panel and its supports/boundaries are assumed different and time independent. An applied static pressure differential exists between the upper and lower surfaces of the panel. And finally, a fluid flows over the upper surface of the panel at supersonic speeds ($3 \leq M \leq 5$) which induces unsteady aerodynamic forces on the panel. The aerodynamic forces are superimposed on the static pressure differential. Structural damping is included to model the energy absorption characteristics of the panel.

The equations of motion are derived for both isotropic and orthotropic panels. Portions of the isotropic equations have been derived in other documented efforts. The isotropic equations were derived in Reference 12 without the applied static loads, initial panel shape, and structural damping. That report stemmed from research performed by the second author and published in Reference 4 which included the initial panel shape and thermal loads. This and future efforts will extend and validate their research.

In Section II, the orthotropic constitutive relations are presented and the strain-displacement relations are derived. These relations are used in the development of the nonlinear equations of motion for panels with slight initial curvature. The nonlinearity is introduced in the relationships between strain and plate deformation which are summarized at the end of Section II.

Section III develops the equations of motion for an orthotropic panel subject to some unspecified nonconservative forces. The strain-displacement and constitutive relations from the previous section are used to define the total strain and bending energies in the panel. Rayleigh-Ritz approximate modes are substituted into the energy expressions which are then nondimensionalized. The nondimensional energy expressions are substituted into a nondimensional form of Lagrange's equation which yields an equation of motion in each coordinate direction.

The isotropic equations of motion are quickly summarized in Section IV following the orthotropic development in Section III.

A brief discussion of some possible transverse acting nonconservative forces is provided in Section V. Here first order piston theory aerodynamics are combined with a static pressure differential and a structural damping model to form the nonconservative forces acting on the

panel. Other aerodynamic theories and structural damping models could be substituted in their place. Transverse forces, other than those specified, also could be incorporated here.

Section VI presents a derivation of piston theory starting from the conservation of momentum and mass equations. The assumptions made in deriving the theory and its limitations are stated. First, second, and third order piston theory expressions are derived by expanding the large amplitude simple wave pressure relation. A fourth piston theory expression is stated that expands the range for which piston theory can be applied.

SECTION II

Stress/Strain/Displacement Relations

The stress-strain and strain-displacement relations are required to derive the equations of motion of a vibrating plate. The four basic assumptions in this section are: (1) the material is orthotropic with its principle axes aligned with the coordinated system, (2) the plate deformations do not exceed the proportional limits, (3) sections that are plane and perpendicular to the median plane prior to loading remain so after loading, and (4) the material properties are independent of temperature.

The constitutive relationship for an orthotropic material in a three dimensional stress state [Ref. 6] is

$$\begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_z \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \frac{1}{E_x} & \frac{-\nu_{xy}}{E_x} & \frac{-\nu_{xz}}{E_x} & 0 & 0 & 0 \\ \frac{-\nu_{xy}}{E_x} & \frac{1}{E_y} & \frac{-\nu_{yz}}{E_y} & 0 & 0 & 0 \\ \frac{-\nu_{xz}}{E_x} & \frac{-\nu_{yz}}{E_y} & \frac{1}{E_z} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G_{yz}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G_{xz}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G_{xy}} \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{Bmatrix} \tag{1}$$

A thermal strain develops within the material when a uniform temperature differential exists between one point of the material and another (or between the panel and its boundaries). The thermal strain is proportional to the coefficient of thermal expansion for the material and the temperature differential,

$$\epsilon_j = \alpha_j \Delta T \tag{2}$$

where $j = x$, y, and z. These thermal strains are added to the right-hand side of Equation (1). Note, there are no thermal shear strains.

The combination of Equations (1) and (2) is used to form the equations of motion of a heated structure when its physical dimensions are of the same order. This analysis, however, examines thin plates, i.e., plates whose thickness is much smaller than its length and width ($h \ll a$ and $h \ll b$). Therefore, the plate can be assumed to be in a state of plane stress where σ_z, τ_{yz}, and τ_{xz} are neglected. This assumption is valid if the upper and lower surfaces are

assumed free of stress [Ref. 2] and that the compression strain perpendicular to the median plane is small compared to the normal strains parallel to the median plane [Ref. 11]. The resulting two dimensional stress-strain relations, written as functions of the strains, are

$$\sigma_x = \frac{1}{\left(1-\nu_{xy}^2 E_y/E_x\right)}\left[E_x\epsilon_x + \nu_{xy}E_y\epsilon_y - (E_x\alpha_x + \nu_{xy}E_y\alpha_y)\Delta T\right] \tag{3.1}$$

$$\sigma_y = \frac{1}{\left(1-\nu_{xy}^2 E_y/E_x\right)}\left[E_y\epsilon_y + \nu_{xy}E_y\epsilon_x - (E_y\alpha_y + \nu_{xy}E_y\alpha_x)\Delta T\right] \tag{3.2}$$

$$\tau_{xy} = G_{xy}\gamma_{xy} \tag{3.3}$$

The stresses are related to the displacements of the plate through the strain-displacement and constitutive relations. The derivation of the strain-displacement relations follows that of Vol'mir [Ref. 11] and Timoshenko [Ref. 10]. The normal and shear strain-displacement relations are developed for specific applied conditions and then all the relations are combined at the end of this section. The displacements in the x, y, and z directions are defined by the functions u(x, y, t), v(x, y, t), and w(x, y, t) respectively. Note that the displacement functions are dependent on time and the location on the panel.

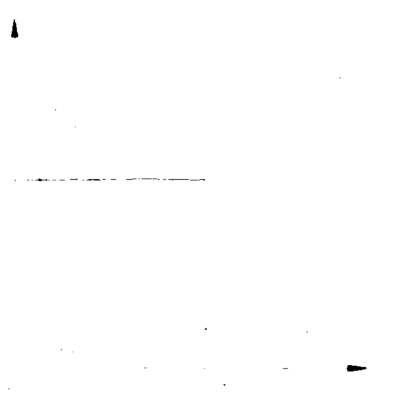

Figure 1 Midplane element undeflected and deflected in the x and y directions.

Figure 1 shows an element (A, B, C, D) of the midplane of the plate before and after deflections are applied in the x-y plane. Table 1 lists the coordinates of points A, A_1, B, B_1, D, and D_1 where the subscript 1 denotes the corners of the deformed element.

Table 1. Coordinates of three corners of undeflected and deflected in-plane element.

$A(0,0)$	$A_1(\mathrm{u},\mathrm{v})$
$B(dx,0)$	$B_1(\mathrm{u}+dx+\frac{\partial \mathrm{u}}{\partial x}dx,\ \mathrm{v}+\frac{\partial \mathrm{v}}{\partial x}dx)$
$D(0,dy)$	$D_1(\mathrm{u}+\frac{\partial \mathrm{u}}{\partial y}dy,\ \mathrm{v}+dy+\frac{\partial \mathrm{v}}{\partial y}dy)$

The normal strain in the x direction is found by examining side $\overline{AB}$. The infinitesimal length before the applied deflection is dx. The length afterwards is

$$\overline{A_1B_1} = \left[\left(dx + \frac{\partial \mathrm{u}}{\partial x}dx\right)^2 + \left(\frac{\partial \mathrm{v}}{\partial x}dx\right)^2\right]^{1/2}$$

$$= dx\left[\left(1 + \frac{\partial \mathrm{u}}{\partial x}\right)^2 + \left(\frac{\partial \mathrm{v}}{\partial x}\right)^2\right]^{1/2}$$

A binomial series expansion of the expanded square root term yields

$$\left[1+2\frac{\partial u}{\partial x}+\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2\right]^{1/2} = 1+\frac{\partial u}{\partial x}+\frac{1}{2}\left(\frac{\partial v}{\partial x}\right)^2+\dots$$

If the cubic and higher order terms are neglected, then the relative strain in the x direction is

$$\epsilon_x^{\prime} = \frac{\overline{A_1B_1}-\overline{AB}}{\overline{AB}} = \frac{\partial u}{\partial x}+\frac{1}{2}\left(\frac{\partial v}{\partial x}\right)^2$$

Similarly, the expression for the relative strain in the y direction is

$$\epsilon_y^{\prime} = \frac{\partial v}{\partial y}+\frac{1}{2}\left(\frac{\partial u}{\partial y}\right)^2$$

If u and v are assumed small compared to w, then their derivatives are also small and the squares of the derivatives are smaller yet. Thus, the squared terms are neglected also. The remaining expressions define the relative normal strain in the x and y directions

$$\epsilon_x^{\prime} = \frac{\partial u}{\partial x} \qquad \epsilon_y^{\prime} = \frac{\partial v}{\partial y} \tag{4}$$

Shear strain is a measure of the deformation of a cubic element into an oblique parallelepiped. It is defined as the angle through which a side of the element is rotated when shearing forces are applied. The shear in the midplane element in Figure 1 is the sum of the two angles: θ, the angle between $\overline{A_1B_1}$ and the x axis, and ϕ, the angle between $\overline{A_1D_1}$ and the y axis. The angles are written in terms of the deflections as

$$\theta = \tan^{-1}\left[\frac{\frac{\partial v}{\partial x}}{1+\frac{\partial u}{\partial x}}\right] \qquad \phi = \tan^{-1}\left[\frac{\frac{\partial u}{\partial y}}{1+\frac{\partial v}{\partial y}}\right]$$

Multiplying the numerator and denominator of θ by $\left(1-\frac{\partial u}{\partial x}\right)$ and ϕ by $\left(1-\frac{\partial v}{\partial y}\right)$ and neglecting the squared partial terms yields

$$\theta = \tan^{-1}\left(\frac{\partial v}{\partial x}\right) \qquad \phi = \tan^{-1}\left(\frac{\partial u}{\partial y}\right)$$

θ and ϕ rewritten using trigonometric series expansions are

$$\theta = \frac{\partial v}{\partial x}-\frac{1}{3}\left(\frac{\partial v}{\partial x}\right)^3+\dots \qquad \phi = \frac{\partial u}{\partial y}-\frac{1}{3}\left(\frac{\partial u}{\partial y}\right)^3+\dots$$

Only the first term from each expansion is retained because the other terms are much smaller in comparison. Thus, the relative shearing strain due to u and v is

$$\gamma_{xy}^{\prime} = \frac{\partial u}{\partial y}+\frac{\partial v}{\partial x} \tag{5}$$

Forces applied in the z direction cause the plate to bend, therefore, bending strain expressions must be included. Figure 2 shows a schematic of a midplane element before and after point C is deflected in the z direction. In this case the plate does not move within the x-y plane. The curvature of the element in a general direction must be defined in order to derive

the bending strain expression. The slope at B_2 relative to A along the x axis is defined using the above trigonometric series expansion,

$$m_x = \frac{\partial w}{\partial x} - \frac{1}{3}\left(\frac{\partial w}{\partial x}\right)^3 \qquad \text{(a)}$$

the slope at D_2 relative to A along the y axis is

$$m_y = \frac{\partial w}{\partial y} - \frac{1}{3}\left(\frac{\partial w}{\partial y}\right)^3 \qquad \text{(b)}$$

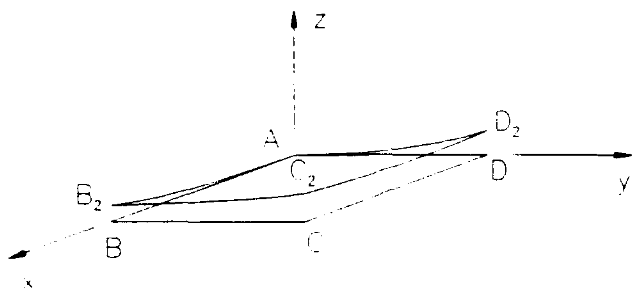

Figure 2 Midplane element in equilibrium and in bending.

and the slope at C_2 relative to A along an arbitrary axis n is

$$m_n = m_x \frac{\partial x}{\partial n} + m_y \frac{\partial y}{\partial n}$$

or

$$m_n = m_x \cos\eta + m_y \sin\eta \qquad \text{(c)}$$

where η is the angle between the x and n axes.

The local curvature is defined negative if the point of evaluation, A, is deflected convex downward as shown in the above figure. (A cup in this orientation will hold water.) The general curvature, K_n, of the middle surface is defined

$$K_n \doteq -\frac{\partial(m_n)}{\partial n} \qquad \text{(d)}$$

where by the chain rule

$$\frac{\partial}{\partial n} = \frac{\partial}{\partial x}\cos\eta + \frac{\partial}{\partial y}\sin\eta$$

The minus sign was introduced in Equation (d) because the second derivative of w with respect to n is negative for a plate with positive curvature. Substituting the derivative expression and (c) into (d) produces

$$K_n = -\left[\frac{\partial m_x}{\partial x}\cos^2\eta + \frac{\partial m_y}{\partial y}\sin^2\eta + \left(\frac{\partial m_y}{\partial x} + \frac{\partial m_x}{\partial y}\right)\sin\eta\cos\eta\right]$$

And finally, substituting in the slope expressions, (a) and (b), gives an expression for the curvature along n

$$K_n = -\frac{\partial^2 w}{\partial x^2}\left[1 + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right]\cos^2\eta - \frac{\partial^2 w}{\partial y^2}\left[1 + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2\right]\sin^2\eta$$

$$- \frac{\partial^2 w}{\partial x \partial y}\left[1 + \frac{1}{4}\left(\frac{\partial w}{\partial x}\right)^2 + \frac{1}{4}\left(\frac{\partial w}{\partial y}\right)^2\right]\sin 2\eta$$

The x, y, and xy curvatures are extracted from this equation. The squared terms within the brackets are neglected because they are two orders of magnitude smaller than the other terms. The curvatures are thus defined as

$$K_x = -\frac{\partial^2 w}{\partial x^2} \qquad K_y = -\frac{\partial^2 w}{\partial y^2} \qquad K_{xy} = \frac{\partial^2 w}{\partial x \partial y} \tag{e,f,g}$$

The strain in the plate due to bending is proportional to the curvature and the distance from the median plane as

$$\epsilon_i = C z K_i \tag{h}$$

where C is 1.0 when i is x or y and C is -2.0 when i is xy. Substituting Equations (e), (f), and (g) into (h) yields the kinematic relation between the out-of-plane deformation and strain

$$\epsilon_x^{//} = -z\frac{\partial^2 w}{\partial x^2} \qquad \epsilon_y^{//} = -z\frac{\partial^2 w}{\partial y^2} \qquad \gamma_{xy}^{//} = -2z\frac{\partial^2 w}{\partial x \partial y} \tag{6}$$

The midplane of the plate stretches when there are large out-of-plane deflections.[1] In this case, the new length of $\overline{AB}$ in Figure 2 is

$$\overline{AB_2} = \left[(dx)^2 + \left(\frac{\partial w}{\partial x}dx\right)^2\right]^{1/2}$$

After factoring out dx and replacing the square root with the first two terms of its binomial series expansion, $\overline{AB_2}$ is

$$\overline{AB_2} \approx dx\left[1 + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right]$$

The relative normal strain in the midplane in the x direction due to large w is thus

$$\epsilon_x^{///} = \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 \tag{7.1}$$

The expression for the relative normal strain in the y direction due to large w is derived in the same manner to be

$$\epsilon_y^{///} = \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 \tag{7.2}$$

The shear strain due to large z deflection is defined by examining the length of each side of the deflected plate. According to Vol'mir

$$\overline{B_2D_2} = \left[(dx)^2 + (dy)^2 + \left(\frac{\partial w}{\partial y}dy - \frac{\partial w}{\partial x}dx\right)^2\right]^{1/2}$$

Applying the cosine law introduces the included angle

$$\left(\overline{B_2D_2}\right)^2 = \left(\overline{AB_2}\right)^2 + \left(\overline{AD_2}\right)^2 - 2\,\overline{AB_2}\,\overline{AD_2}\cos\left(\frac{\pi}{2} - \gamma_{xy}^{///}\right)$$

and replacing the cosine term with the first term of its series expansion yields

[1] Deflections are limited such that the curvature is less than 0.12 radians per longest panel length. Expressions (e), (f), and (g) must include the neglected terms if the curvature is greater than this value.

$$\left(\overline{B_2 D_2}\right)^2 \approx \left(\overline{AB_2}\right)^2 + \left(\overline{AD_2}\right)^2 - 2\,\overline{AB_2}\,\overline{AD_2}\,\gamma_{xy}{}^{\prime\prime\prime}$$

Substituting in the appropriate length expressions and neglecting the squared derivatives on the left-hand side results in an expression for the relative shear strain in the midplane due to large out-of-plane displacement,

$$\gamma_{xy}{}^{\prime\prime\prime} = \frac{\partial \mathrm{w}}{\partial x}\frac{\partial \mathrm{w}}{\partial y} \tag{7.3}$$

If the plate is not flat before any load is applied, then the initial shape of the plate needs to be included in the strain-displacement relations. z_p is a function of x and y that describes the initial shape of the midplane of the plate, and w is the deflection in the z direction measured from z_p. It is assumed that the maximum value of z_p is less than or equal to the plate thickness. The total deflection from the x-y plane is written

$$\mathrm{w_T} = z_p + \mathrm{w}$$

The strains in the midplane are derived as before. The initial length of side $\overline{AB}$ is

$$\overline{AB} = dx\left[1 + \left(\frac{\partial z_p}{\partial x}\right)^2\right]^{1/2}$$

Using a binomial series expansion again yields

$$\overline{AB} \approx dx\left[1 + \frac{1}{2}\left(\frac{\partial z_p}{\partial x}\right)^2\right]$$

and the length after the plate deflection is

$$\overline{AB_3} \approx dx\left[1 + \frac{1}{2}\left(\frac{\partial \mathrm{w_T}}{\partial x}\right)^2\right] = dx\left[1 + \frac{1}{2}\left(\frac{\partial z_p}{\partial x}\right)^2 + \frac{\partial z_p}{\partial x}\frac{\partial \mathrm{w}}{\partial x} + \frac{1}{2}\left(\frac{\partial \mathrm{w}}{\partial x}\right)^2\right]$$

The relative strain in the x direction due to a deflection w above a non-flat plate is

$$\epsilon_x{}^{iv} = \frac{\overline{AB_3} - \overline{AB}}{\overline{AB}} = \frac{\dfrac{\partial z_p}{\partial x}\dfrac{\partial \mathrm{w}}{\partial x} + \dfrac{1}{2}\left(\dfrac{\partial \mathrm{w}}{\partial x}\right)^2}{1 + \dfrac{1}{2}\left(\dfrac{\partial z_p}{\partial x}\right)^2}$$

Multiplying the numerator and denominator by $1 - \frac{1}{2}\left(\frac{\partial z_p}{\partial x}\right)^2$ and neglecting the fourth order partial derivative terms yields a new strain term in addition to (7.1)

$$\epsilon_x{}^{iv} = \frac{\partial z_p}{\partial x}\frac{\partial \mathrm{w}}{\partial x} \tag{8.1}$$

The relative strain in the y direction under these conditions is

$$\epsilon_y{}^{iv} = \frac{\partial z_p}{\partial y}\frac{\partial \mathrm{w}}{\partial y} \tag{8.2}$$

The shear strain relation is again based on the change in the angle between dx and dy. In this case, the initial and final angles are

$$\theta_i = \frac{\pi}{2} - \frac{\partial z_p}{\partial x}\frac{\partial z_p}{\partial y}$$

$$\theta_f = \frac{\pi}{2} - \frac{\partial w_T}{\partial x}\frac{\partial w_T}{\partial y}$$

The shear strain term due to a deflection w in the z direction above a non-flat plate is

$$\gamma_{xy}{}^{iv} = \frac{\partial z_p}{\partial x}\frac{\partial \mathrm{w}}{\partial y} + \frac{\partial z_p}{\partial y}\frac{\partial \mathrm{w}}{\partial x} \tag{8.3}$$

The strain-displacement relations for a non-flat plate that is being stretched in the x and y directions and bent in the z direction are the sum of the previous strain expressions

$$\epsilon_x = \frac{\partial \mathrm{u}}{\partial x} - z\frac{\partial^2 \mathrm{w}}{\partial x^2} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 + \frac{\partial z_p}{\partial x}\frac{\partial \mathrm{w}}{\partial x} \tag{9.1}$$

$$\epsilon_y = \frac{\partial \mathrm{v}}{\partial y} - z\frac{\partial^2 \mathrm{w}}{\partial y^2} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 + \frac{\partial z_p}{\partial y}\frac{\partial \mathrm{w}}{\partial y} \tag{9.2}$$

$$\gamma_{xy} = \frac{\partial \mathrm{v}}{\partial x} + \frac{\partial \mathrm{u}}{\partial y} + \frac{\partial \mathrm{w}}{\partial x}\frac{\partial \mathrm{w}}{\partial y} - 2z\frac{\partial^2 \mathrm{w}}{\partial x \partial y} + \frac{\partial z_p}{\partial x}\frac{\partial \mathrm{w}}{\partial y} + \frac{\partial z_p}{\partial y}\frac{\partial \mathrm{w}}{\partial x} \tag{9.3}$$

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SECTION III

Orthotropic Equation Development

The equations of motion of an orthotropic panel are derived using Lagrange's equation,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_{ij}}\right) - \frac{\partial L}{\partial q_{ij}} + Q_{ij} = 0 \tag{10}$$

where q_{ij} are the generalized coordinates. The Lagrangian, L, is the difference between the total kinetic energy, T, and the total elastic energy, U. The total linear elastic strain energy in a body is defined as

$$U = \frac{1}{2}\iiint\left[\sigma_x \epsilon_x + \sigma_y \epsilon_y + \sigma_z \epsilon_z + \tau_{xy}\gamma_{xy} + \tau_{yz}\gamma_{yz} + \tau_{zx}\gamma_{zx}\right] dx\,dy\,dz \tag{11}$$

and the total linear kinetic energy is

$$T = \frac{1}{2}\iint m_p\left[\left(\frac{\partial \mathrm{u}}{\partial t}\right)^2 + \left(\frac{\partial \mathrm{v}}{\partial t}\right)^2 + \left(\frac{\partial \mathrm{w}}{\partial t}\right)^2\right] dx\,dy$$

However, since the in-plane displacements, u and v, are assumed small in comparison to w, the kinetic energy will be written as

$$T = \frac{1}{2}\iint m_p\left(\frac{\partial \mathrm{w}}{\partial t}\right)^2 dx\,dy \tag{12}$$

The plane-stress assumption, i.e., the stresses through the thickness are assumed small compared to the in-plane stresses, was introduced in the last section. Incorporating this assumption into the elastic energy expression yields

$$U = \frac{1}{2}\iiint\left[\sigma_x \epsilon_x + \sigma_y \epsilon_y + \tau_{xy}\gamma_{xy}\right] dx\,dy\,dz \tag{13}$$

Applied in-plane loads (no shearing load) are included through the stress-strain relations, (3.1) and (3.2), as

$$\sigma_x = \frac{E_x}{(1-E^*\nu_{xy}^2)}\left[\epsilon_x + E^*\nu_{xy}\epsilon_y - (\alpha_x + E^*\nu_{xy}\alpha_y)\Delta T\right] + \frac{1}{h}N_x^{\,a} \tag{14.1}$$

$$\sigma_y = \frac{E_y}{(1-E^*\nu_{xy}^2)}\left[\epsilon_y + \nu_{xy}\epsilon_x - (\alpha_y + \nu_{xy}\alpha_x)\Delta T\right] + \frac{1}{h}N_y^{\,a} \tag{14.2}$$

The applied loads are positive if the plate is in tension.

Equations (3.3) and (14) are substituted into (13) to produce

$$U = \frac{E_x}{2(1-E^*\nu_{xy}^2)}\iiint\Big[\epsilon_x^2 + E^*\epsilon_y^2 + 2E^*\nu_{xy}\epsilon_x\epsilon_y + (1-E^*\nu_{xy}^2)\frac{G_{xy}}{E_x}\gamma_{xy}^2$$

$$- \left(\epsilon_x\alpha_x + E^*\nu_{xy}\epsilon_x\alpha_y + E^*\epsilon_y\alpha_y + E^*\nu_{xy}\epsilon_y\alpha_x\right)\Delta T(x,y)$$

$$+(1-E^*\nu_{xy}^2)\frac{1}{E_x h}\left(N_x^a\epsilon_x+N_y^a\epsilon_y\right)\Big]\,dx\,dy\,dz \tag{15}$$

Note that the temperature differential with respect to the panel is a function of x and y.

The strain-displacement relations, Equations (9), are rewritten as

$$\epsilon_x = \epsilon_{xo} - z\frac{\partial^2 w}{\partial x^2} \qquad \epsilon_y = \epsilon_{yo} - z\frac{\partial^2 w}{\partial y^2} \qquad \gamma_{xy} = \gamma_{xyo} - 2z\frac{\partial^2 w}{\partial x\partial y} \tag{16}$$

where ϵ_{xo}, ϵ_{yo}, and γ_{xyo} are the stretching strain terms. Substituting (16) into (15) gives

$$\begin{aligned}
U = \frac{E_x}{2(1-E^*\nu_{xy}^2)}\iiint\Bigg\{&\epsilon_{xo}^2+E^*\epsilon_{yo}^2+2E^*\nu_{xy}\epsilon_{xo}\epsilon_{yo}+(1-E^*\nu_{xy}^2)\frac{G_{xy}}{E_x}\gamma_{xyo}^2\\
&-\left(\epsilon_{xo}+E^*\nu_{xy}\frac{\alpha_y}{\alpha_x}\epsilon_{xo}+E^*\frac{\alpha_y}{\alpha_x}\epsilon_{yo}+E^*\nu_{xy}\epsilon_{yo}\right)\alpha_x\Delta T(x,y)\\
&+\frac{(1-E^*\nu_{xy}^2)}{E_x h}\left(N_x^a\epsilon_{xo}+N_y^a\epsilon_{yo}\right)\\
&+z^2\Bigg[\left(\frac{\partial^2 w}{\partial x^2}\right)^2+E^*\left(\frac{\partial^2 w}{\partial y^2}\right)^2+2E^*\nu_{xy}\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}\\
&\qquad+4(1-E^*\nu_{xy}^2)\frac{G_{xy}}{E_x}\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2\Bigg]\\
&-2z\Bigg[\epsilon_{xo}\frac{\partial^2 w}{\partial x^2}+E^*\epsilon_{yo}\frac{\partial^2 w}{\partial y^2}+E^*\nu_{xy}\left(\epsilon_{xo}\frac{\partial^2 w}{\partial y^2}+\epsilon_{yo}\frac{\partial^2 w}{\partial x^2}\right)\\
&\qquad+2(1-E^*\nu_{xy}^2)\frac{G_{xy}}{E_x}\gamma_{xyo}\frac{\partial^2 w}{\partial x\partial y}+\frac{1}{2}\Bigg(\frac{\partial^2 w}{\partial x^2}\\
&\qquad+E^*\nu_{xy}\frac{\alpha_y}{\alpha_x}\frac{\partial^2 w}{\partial x^2}+E^*\frac{\alpha_y}{\alpha_x}\frac{\partial^2 w}{\partial y^2}+E^*\nu_{xy}\frac{\partial^2 w}{\partial y^2}\Bigg)\alpha_x\Delta T(x,y)\\
&\qquad+\frac{(1-E^*\nu_{xy}^2)}{2E_x h}\left(N_x^a\frac{\partial^2 w}{\partial x^2}+N_y^a\frac{\partial^2 w}{\partial y^2}\right)\Bigg]\Bigg\}\,dx\,dy\,dz
\end{aligned}$$

The result after integrating with respect to z from $-\frac{h}{2}$ to $\frac{h}{2}$ is

$$\begin{aligned}
U = \frac{E_x h}{2(1-E^*\nu_{xy}^2)}\iint\Bigg[&\epsilon_{xo}^2+E^*\epsilon_{yo}^2+2E^*\nu_{xy}\epsilon_{xo}\epsilon_{yo}+(1-E^*\nu_{xy}^2)\frac{G_{xy}}{E_x}\gamma_{xyo}^2\\
&-\left(\epsilon_{xo}+E^*\nu_{xy}\frac{\alpha_y}{\alpha_x}\epsilon_{xo}+E^*\frac{\alpha_y}{\alpha_x}\epsilon_{yo}+E^*\nu_{xy}\epsilon_{yo}\right)\alpha_x\Delta T(x,y)
\end{aligned}$$

$$+\frac{(1-E^{*}\nu_{xy}^{2})}{E_{x}h}\left(N_{x}^{a}\epsilon_{xo}+N_{y}^{a}\epsilon_{yo}\right)\Bigg]\,dx\,dy$$

$$+\frac{D_{x}}{2}\iint\left[\left(\frac{\partial^{2}w}{\partial x^{2}}\right)^{2}+E^{*}\left(\frac{\partial^{2}w}{\partial y^{2}}\right)^{2}+2E^{*}\nu_{xy}\frac{\partial^{2}w}{\partial x^{2}}\frac{\partial^{2}w}{\partial y^{2}}\right.$$

$$\left.+4(1-E^{*}\nu_{xy}^{2})\frac{G_{xy}}{E_{x}}\left(\frac{\partial^{2}w}{\partial x\,\partial y}\right)^{2}\right]dx\,dy \tag{17}$$

where

$$D_{x}=\frac{E_{x}h^{3}}{12(1-E^{*}\nu_{xy}^{2})}$$

The first double integral is the total stretching strain energy, U_S, and the second is the total bending strain energy, U_B. They are restated here with the stretching strain terms ıeplaced with their respective strain-displacement relations

$$U_{S}=\frac{E_{x}h}{2(1-E^{*}\nu_{xy}^{2})}\iint\left\{\left[\frac{\partial u}{\partial x}+\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^{2}+\frac{\partial z_{p}}{\partial x}\frac{\partial w}{\partial x}\right]^{2}+E^{*}\left[\frac{\partial v}{\partial y}+\frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^{2}+\frac{\partial z_{p}}{\partial y}\frac{\partial w}{\partial y}\right]^{2}\right.$$

$$+2E^{*}\nu_{xy}\left[\frac{\partial u}{\partial x}+\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^{2}+\frac{\partial z_{p}}{\partial x}\frac{\partial w}{\partial x}\right]\left[\frac{\partial v}{\partial y}+\frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^{2}+\frac{\partial z_{p}}{\partial y}\frac{\partial w}{\partial y}\right]$$

$$+(1-E^{*}\nu_{xy}^{2})\frac{G_{xy}}{E_{x}}\left[\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}+\frac{\partial z_{p}}{\partial x}\frac{\partial w}{\partial y}+\frac{\partial z_{p}}{\partial y}\frac{\partial w}{\partial x}\right]^{2}$$

$$-\left(1+E^{*}\nu_{xy}\frac{\alpha_{y}}{\alpha_{x}}\right)\left[\frac{\partial u}{\partial x}+\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^{2}+\frac{\partial z_{p}}{\partial x}\frac{\partial w}{\partial x}\right]\alpha_{x}\Delta T(x,y)$$

$$-E^{*}\left(\frac{\alpha_{y}}{\alpha_{x}}+\nu_{xy}\right)\left[\frac{\partial v}{\partial y}+\frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^{2}+\frac{\partial z_{p}}{\partial y}\frac{\partial w}{\partial y}\right]\alpha_{x}\Delta T(x,y)$$

$$+\frac{(1-E^{*}\nu_{xy}^{2})}{E_{x}h}\left[\frac{\partial u}{\partial x}+\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^{2}+\frac{\partial z_{p}}{\partial x}\frac{\partial w}{\partial x}\right]N_{x}^{a}$$

$$\left.+\frac{(1-E^{*}\nu_{xy}^{2})}{E_{x}h}\left[\frac{\partial v}{\partial y}+\frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^{2}+\frac{\partial z_{p}}{\partial y}\frac{\partial w}{\partial y}\right]N_{y}^{a}\right\}dx\,dy \tag{18}$$

and

$$U_{B}=\frac{D_{x}}{2}\iint\left[\left(\frac{\partial^{2}w}{\partial x^{2}}\right)^{2}+E^{*}\left(\frac{\partial^{2}w}{\partial y^{2}}\right)^{2}+2E^{*}\nu_{xy}\frac{\partial^{2}w}{\partial x^{2}}\frac{\partial^{2}w}{\partial y^{2}}\right.$$

$$\left.+4(1-E^{*}\nu_{xy}^{2})\frac{G_{xy}}{E_{x}}\left(\frac{\partial^{2}w}{\partial x\,\partial y}\right)^{2}\right]dx\,dy \tag{19}$$

Note at this point that the applied conservative loads do not contribute to the bending strain energy. However, they would if the panel thickness was asymmetric about the median

plane.

A Rayleigh-Ritz approach is used to represent the three displacements, u, v, and w, with the product of temporal and spacial (modal) functions

$$
\begin{aligned}
\mathrm{u} &= a_{ij}(t)\,\alpha_i(x)\,\beta_j(y) \\
\mathrm{v} &= b_{kl}(t)\,\alpha_k(x)\,\beta_l(y) \\
\mathrm{w} &= c_{mn}(t)\,\phi_m(x)\,\psi_n(y)
\end{aligned}
\tag{20}
$$

where the modal functions are dependent upon the plate boundary conditions only and the indices are summed over the number of modes included in the analysis. The temporal functions are the generalized coordinates.

Equations (10), (12), (18), (19), and (20) are sufficient to define the dimensional equations of motion of a plate that is subject to the specified conservative loads and some unspecified nonconservative load, Q_{ij}. However, the nondimensional equations of motion are desired. The following nondimensional parameters are substituted into the above equations.

$$
\begin{array}{llllll}
a_{ij} = h\bar{a}_{ij} & b_{kl} = h\bar{b}_{kl} & c_{mn} = h\bar{c}_{mn} & x = a\bar{x} & y = b\bar{y} & z = h\bar{z} \\
\mathrm{u} = h\bar{\mathrm{u}} & \mathrm{v} = h\bar{\mathrm{v}} & \mathrm{w} = h\bar{\mathrm{w}} & t = \tau\left(\dfrac{m_p a^4}{D_x}\right)^{1/2} & Q_{ij} = \dfrac{1}{h}\bar{Q}_{ij} & z_p = h\bar{z}_p
\end{array}
\tag{21}
$$

Lagrange's equation becomes

$$
\frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{\bar{q}}_{ij}}\right) - \frac{\partial L}{\partial \bar{q}_{ij}} + \bar{Q}_{ij} = 0 \tag{22}
$$

the total kinetic energy is rewritten as

$$
T = \frac{D_x}{2}\frac{b}{a}\left(\frac{h}{a}\right)^2 \int\!\!\int \left(\frac{\partial \bar{\mathrm{w}}}{\partial \tau}\right)^2 d\bar{x}\,d\bar{y} \tag{23}
$$

the total bending strain energy is now

$$
\begin{aligned}
U_B = \frac{D_x}{2}\frac{b}{a}\left(\frac{h}{a}\right)^2 \int\!\!\int \Bigg[&\left(\frac{\partial^2 \bar{\mathrm{w}}}{\partial \bar{x}^2}\right)^2 + E^*\left(\frac{a}{b}\right)^4\left(\frac{\partial^2 \bar{\mathrm{w}}}{\partial \bar{y}^2}\right)^2 + 2E^*\nu_{xy}\left(\frac{a}{b}\right)^2 \frac{\partial^2 \bar{\mathrm{w}}}{\partial \bar{x}^2}\frac{\partial^2 \bar{\mathrm{w}}}{\partial \bar{y}^2} \\
&+ 4D_{xy}\left(\frac{a}{b}\right)^2\left(\frac{\partial^2 \bar{\mathrm{w}}}{\partial \bar{x}\,\partial \bar{y}}\right)^2 \Bigg] d\bar{x}\,d\bar{y}
\end{aligned}
\tag{24}
$$

where $D_{xy} = \dfrac{G_{xy}}{E_x}(1 - E^*\nu_{xy}^2)$. The total nondimensional stretching strain energy is

$$
\begin{aligned}
U_S = 6D_x\frac{b}{a}\int\!\!\int \Bigg\{ &\left[\frac{\partial \bar{\mathrm{u}}}{\partial \bar{x}} + \frac{1}{2}\frac{h}{a}\left(\frac{\partial \bar{\mathrm{w}}}{\partial \bar{x}}\right)^2 + \frac{h}{a}\frac{\partial \bar{z}_p}{\partial \bar{x}}\frac{\partial \bar{\mathrm{w}}}{\partial \bar{x}}\right]^2 + E^*\left(\frac{a}{b}\right)^2\left[\frac{\partial \bar{\mathrm{v}}}{\partial \bar{y}} + \frac{1}{2}\frac{h}{b}\left(\frac{\partial \bar{\mathrm{w}}}{\partial \bar{y}}\right)^2 + \frac{h}{b}\frac{\partial \bar{z}_p}{\partial \bar{y}}\frac{\partial \bar{\mathrm{w}}}{\partial \bar{y}}\right]^2 \\
&+ 2E^*\nu_{xy}\frac{a}{b}\left[\frac{\partial \bar{\mathrm{u}}}{\partial \bar{x}} + \frac{1}{2}\frac{h}{a}\left(\frac{\partial \bar{\mathrm{w}}}{\partial \bar{x}}\right)^2 + \frac{h}{a}\frac{\partial \bar{z}_p}{\partial \bar{x}}\frac{\partial \bar{\mathrm{w}}}{\partial \bar{x}}\right]\left[\frac{\partial \bar{\mathrm{v}}}{\partial \bar{y}} + \frac{1}{2}\frac{h}{b}\left(\frac{\partial \bar{\mathrm{w}}}{\partial \bar{y}}\right)^2 + \frac{h}{b}\frac{\partial \bar{z}_p}{\partial \bar{y}}\frac{\partial \bar{\mathrm{w}}}{\partial \bar{y}}\right]
\end{aligned}
$$

$$+D_{xy}\left[\frac{\partial\bar{v}}{\partial\bar{x}}+\frac{a}{b}\frac{\partial\bar{u}}{\partial\bar{y}}+\frac{h}{b}\frac{\partial\bar{w}}{\partial\bar{x}}\frac{\partial\bar{w}}{\partial\bar{y}}+\frac{h}{b}\frac{\partial\bar{z}_p}{\partial\bar{x}}\frac{\partial\bar{w}}{\partial\bar{y}}+\frac{h}{b}\frac{\partial\bar{z}_p}{\partial\bar{y}}\frac{\partial\bar{w}}{\partial\bar{x}}\right]^2$$

$$-\left(1+E^*\nu_{xy}\frac{\alpha_y}{\alpha_x}\right)\left[\frac{a}{h}\frac{\partial\bar{u}}{\partial\bar{x}}+\frac{1}{2}\left(\frac{\partial\bar{w}}{\partial\bar{x}}\right)^2+\frac{\partial\bar{z}_p}{\partial\bar{x}}\frac{\partial\bar{w}}{\partial\bar{x}}\right]\alpha_x\,\Delta T(\bar{x},\bar{y})$$

$$-E^*\left(\frac{\alpha_y}{\alpha_x}+\nu_{xy}\right)\frac{a}{b}\left[\frac{a}{h}\frac{\partial\bar{v}}{\partial\bar{y}}+\frac{1}{2}\frac{a}{b}\left(\frac{\partial\bar{w}}{\partial\bar{y}}\right)^2+\frac{a}{b}\frac{\partial\bar{z}_p}{\partial\bar{y}}\frac{\partial\bar{w}}{\partial\bar{y}}\right]\alpha_x\,\Delta T(\bar{x},\bar{y})$$

$$+\frac{\left(1-E^*\nu_{xy}^2\right)}{E_x h}\left[\frac{a}{h}\frac{\partial\bar{u}}{\partial\bar{x}}+\frac{1}{2}\left(\frac{\partial\bar{w}}{\partial\bar{x}}\right)^2+\frac{\partial\bar{z}_p}{\partial\bar{x}}\frac{\partial\bar{w}}{\partial\bar{x}}\right]N_x^a$$

$$+\frac{\left(1-E^*\nu_{xy}^2\right)}{E_x h}\frac{a}{b}\left[\frac{a}{h}\frac{\partial\bar{v}}{\partial\bar{y}}+\frac{1}{2}\frac{a}{b}\left(\frac{\partial\bar{w}}{\partial\bar{y}}\right)^2+\frac{a}{b}\frac{\partial\bar{z}_p}{\partial\bar{y}}\frac{\partial\bar{w}}{\partial\bar{y}}\right]N_y^a\Bigg\}\,d\bar{x}\,d\bar{y} \tag{25}$$

The nondimensional displacement functions are

$$\begin{aligned}\bar{u} &= \bar{a}_{ij}(\tau)\,\alpha_i(\bar{x})\,\beta_j(\bar{y})\\ \bar{v} &= \bar{b}_{kl}(\tau)\,\alpha_k(\bar{x})\,\beta_l(\bar{y})\\ \bar{w} &= \bar{c}_{mn}(\tau)\,\phi_m(\bar{x})\,\psi_n(\bar{y})\end{aligned} \tag{26}$$

The kinetic energy written in terms of the nondimensional displacement functions is

$$T = \frac{D_x}{2}\frac{b}{a}\left(\frac{h}{a}\right)^2\dot{\bar{c}}_{kl}\,\dot{\bar{c}}_{ij}\int\!\!\int\phi_k\phi_i\psi_l\psi_j\,d\bar{x}\,d\bar{y} \tag{27}$$

The bending strain energy written in terms of the displacement functions is

$$\begin{aligned}U_B = \frac{D_x}{2}\frac{b}{a}\left(\frac{h}{a}\right)^2\bar{c}_{kl}\bar{c}_{ij}\int\!\!\int\Bigg[&\frac{\partial^2\phi_k}{\partial\bar{x}^2}\frac{\partial^2\phi_i}{\partial\bar{x}^2}\psi_l\psi_j+E^*\left(\frac{a}{b}\right)^4\phi_k\phi_i\frac{\partial^2\psi_l}{\partial\bar{y}^2}\frac{\partial^2\psi_j}{\partial\bar{y}^2}\\ &+E^*\nu_{xy}\left(\frac{a}{b}\right)^2\frac{\partial^2\phi_k}{\partial\bar{x}^2}\phi_i\psi_l\frac{\partial^2\psi_j}{\partial\bar{y}^2}+E^*\nu_{xy}\left(\frac{a}{b}\right)^2\phi_k\frac{\partial^2\phi_i}{\partial\bar{x}^2}\frac{\partial^2\psi_l}{\partial\bar{y}^2}\psi_j\\ &+4D_{xy}\left(\frac{a}{b}\right)^2\frac{\partial\phi_k}{\partial\bar{x}}\frac{\partial\phi_i}{\partial\bar{x}}\frac{\partial\psi_l}{\partial\bar{y}}\frac{\partial\psi_j}{\partial\bar{y}}\Bigg]d\bar{x}\,d\bar{y}\end{aligned} \tag{28}$$

And finally, the stretching strain energy equation becomes

$$\begin{aligned}U_S = 6D_x\frac{b}{a}\int\!\!\int\Bigg\{&\bar{a}_{kl}\bar{a}_{ij}\frac{\partial\alpha_k}{\partial\bar{x}}\frac{\partial\alpha_i}{\partial\bar{x}}\beta_l\beta_j+\frac{h}{a}\bar{c}_{kl}\bar{c}_{mn}\bar{a}_{ij}\frac{\partial\phi_k}{\partial\bar{x}}\frac{\partial\phi_m}{\partial\bar{x}}\frac{\partial\alpha_i}{\partial\bar{x}}\psi_l\psi_n\beta_j\\ &+\frac{1}{4}\left(\frac{h}{a}\right)^2\bar{c}_{kl}\bar{c}_{mn}\bar{c}_{op}\bar{c}_{ij}\frac{\partial\phi_k}{\partial\bar{x}}\frac{\partial\phi_m}{\partial\bar{x}}\frac{\partial\phi_o}{\partial\bar{x}}\frac{\partial\phi_i}{\partial\bar{x}}\psi_l\psi_n\psi_p\psi_j\\ &+2\frac{h}{a}\bar{c}_{kl}\bar{a}_{ij}\frac{\partial\bar{z}_p}{\partial\bar{x}}\frac{\partial\phi_k}{\partial\bar{x}}\frac{\partial\alpha_i}{\partial\bar{x}}\psi_l\beta_j+\left(\frac{h}{a}\right)^2\bar{c}_{kl}\bar{c}_{ij}\left(\frac{\partial\bar{z}_p}{\partial\bar{x}}\right)^2\frac{\partial\phi_k}{\partial\bar{x}}\frac{\partial\phi_i}{\partial\bar{x}}\psi_l\psi_j\end{aligned}$$

$$
\begin{aligned}
&+\left(\frac{h}{a}\right)^2 \bar{c}_{kl}\,\bar{c}_{mn}\,\bar{c}_{ij}\,\frac{\partial \bar{z}_p}{\partial \bar{x}}\,\frac{\partial \phi_k}{\partial \bar{x}}\,\frac{\partial \phi_m}{\partial \bar{x}}\,\frac{\partial \phi_i}{\partial \bar{x}}\,\psi_l \psi_n \psi_j \\
&+E^*\left(\frac{a}{b}\right)^2\Bigg[\bar{b}_{kl}\,\bar{b}_{ij}\,\alpha_k \alpha_i\,\frac{\partial \beta_l}{\partial \bar{y}}\,\frac{\partial \beta_j}{\partial \bar{y}} + \frac{h}{b}\,\bar{c}_{kl}\,\bar{c}_{mn}\,\bar{b}_{ij}\,\phi_k \phi_m \alpha_i\,\frac{\partial \psi_l}{\partial \bar{y}}\,\frac{\partial \psi_n}{\partial \bar{y}}\,\frac{\partial \beta_j}{\partial \bar{y}} \\
&\qquad + \frac{1}{4}\left(\frac{h}{b}\right)^2 \bar{c}_{kl}\,\bar{c}_{mn}\,\bar{c}_{op}\,\bar{c}_{ij}\,\phi_k \phi_m \phi_o \phi_i\,\frac{\partial \psi_l}{\partial \bar{y}}\,\frac{\partial \psi_n}{\partial \bar{y}}\,\frac{\partial \psi_p}{\partial \bar{y}}\,\frac{\partial \psi_j}{\partial \bar{y}} \\
&\qquad + 2\,\frac{h}{b}\,\bar{c}_{kl}\,\bar{b}_{ij}\,\frac{\partial \bar{z}_p}{\partial \bar{y}}\,\phi_k \alpha_i\,\frac{\partial \psi_l}{\partial \bar{y}}\,\frac{\partial \beta_j}{\partial \bar{y}} \\
&\qquad + \left(\frac{h}{b}\right)^2 \bar{c}_{kl}\,\bar{c}_{ij}\left(\frac{\partial \bar{z}_p}{\partial \bar{y}}\right)^2 \phi_k \phi_i\,\frac{\partial \psi_l}{\partial \bar{y}}\,\frac{\partial \psi_j}{\partial \bar{y}} \\
&\qquad + \left(\frac{h}{b}\right)^2 \bar{c}_{kl}\,\bar{c}_{mn}\,\bar{c}_{ij}\,\frac{\partial \bar{z}_p}{\partial \bar{y}}\,\phi_k \phi_m \phi_i\,\frac{\partial \psi_l}{\partial \bar{y}}\,\frac{\partial \psi_n}{\partial \bar{y}}\,\frac{\partial \psi_j}{\partial \bar{y}}\Bigg] \\
&+2E^*\nu_{xy}\,\frac{a}{b}\Bigg[\bar{b}_{kl}\,\bar{a}_{ij}\,\alpha_k\,\frac{\partial \alpha_i}{\partial \bar{x}}\,\frac{\partial \beta_l}{\partial \bar{y}}\,\beta_j + \frac{1}{2}\,\frac{h}{b}\,\bar{c}_{kl}\,\bar{c}_{mn}\,\bar{a}_{ij}\,\phi_k \phi_m\,\frac{\partial \alpha_i}{\partial \bar{x}}\,\frac{\partial \psi_l}{\partial \bar{y}}\,\frac{\partial \psi_n}{\partial \bar{y}}\,\beta_j \\
&\qquad + \frac{1}{2}\,\frac{h}{a}\,\bar{c}_{kl}\,\bar{c}_{mn}\,\bar{b}_{ij}\,\frac{\partial \phi_k}{\partial \bar{x}}\,\frac{\partial \phi_m}{\partial \bar{x}}\,\alpha_i \psi_l \psi_n\,\frac{\partial \beta_j}{\partial \bar{y}} \\
&\qquad + \frac{1}{4}\,\frac{h^2}{ab}\,\bar{c}_{kl}\,\bar{c}_{mn}\,\bar{c}_{op}\,\bar{c}_{ij}\,\frac{\partial \phi_k}{\partial \bar{x}}\,\phi_m \phi_o\,\frac{\partial \phi_i}{\partial \bar{x}}\,\psi_l\,\frac{\partial \psi_n}{\partial \bar{y}}\,\frac{\partial \psi_p}{\partial \bar{y}}\,\psi_j \\
&\qquad + \frac{h}{b}\,\bar{c}_{kl}\,\bar{a}_{ij}\,\frac{\partial \bar{z}_p}{\partial \bar{y}}\,\phi_k\,\frac{\partial \alpha_i}{\partial \bar{x}}\,\frac{\partial \psi_l}{\partial \bar{y}}\,\beta_j + \frac{h}{a}\,\bar{c}_{kl}\,\bar{b}_{ij}\,\frac{\partial \bar{z}_p}{\partial \bar{x}}\,\frac{\partial \phi_k}{\partial \bar{x}}\,\alpha_i \psi_l\,\frac{\partial \beta_j}{\partial \bar{y}} \\
&\qquad + \frac{1}{2}\,\frac{h^2}{ab}\,\bar{c}_{kl}\,\bar{c}_{mn}\,\bar{c}_{ij}\,\frac{\partial \bar{z}_p}{\partial \bar{y}}\,\frac{\partial \phi_k}{\partial \bar{x}}\,\phi_m\,\frac{\partial \phi_i}{\partial \bar{x}}\,\psi_l\,\frac{\partial \psi_n}{\partial \bar{y}}\,\psi_j \\
&\qquad + \frac{1}{2}\,\frac{h^2}{ab}\,\bar{c}_{kl}\,\bar{c}_{mn}\,\bar{c}_{ij}\,\frac{\partial \bar{z}_p}{\partial \bar{x}}\,\phi_k\,\frac{\partial \phi_m}{\partial \bar{x}}\,\phi_i\,\frac{\partial \psi_l}{\partial \bar{y}}\,\psi_n\,\frac{\partial \psi_j}{\partial \bar{y}} \\
&\qquad + \frac{h^2}{ab}\,\bar{c}_{kl}\,\bar{c}_{ij}\,\frac{\partial \bar{z}_p}{\partial \bar{x}}\,\frac{\partial \bar{z}_p}{\partial \bar{y}}\,\frac{\partial \phi_k}{\partial \bar{x}}\,\phi_i \psi_l\,\frac{\partial \psi_j}{\partial \bar{y}}\Bigg] \\
&+D_{xy}\Bigg[\bar{b}_{kl}\,\bar{b}_{ij}\,\frac{\partial \alpha_k}{\partial \bar{x}}\,\frac{\partial \alpha_i}{\partial \bar{x}}\,\beta_l \beta_j + \left(\frac{a}{b}\right)^2 \bar{a}_{kl}\,\bar{a}_{ij}\,\alpha_k \alpha_i\,\frac{\partial \beta_l}{\partial \bar{y}}\,\frac{\partial \beta_j}{\partial \bar{y}} \\
&\qquad + 2\,\frac{a}{b}\,\bar{b}_{kl}\,\bar{a}_{ij}\,\frac{\partial \alpha_k}{\partial \bar{x}}\,\alpha_i \beta_l\,\frac{\partial \beta_j}{\partial \bar{y}} + 2\,\frac{h}{b}\,\bar{c}_{kl}\,\bar{c}_{mn}\,\bar{b}_{ij}\,\frac{\partial \phi_k}{\partial \bar{x}}\,\phi_m\,\frac{\partial \alpha_i}{\partial \bar{x}}\,\psi_l\,\frac{\partial \psi_n}{\partial \bar{y}}\,\beta_j \\
&\qquad + 2\,\frac{a}{b}\,\frac{h}{b}\,\bar{c}_{kl}\,\bar{c}_{mn}\,\bar{a}_{ij}\,\phi_k\,\frac{\partial \phi_m}{\partial \bar{x}}\,\alpha_i\,\frac{\partial \psi_l}{\partial \bar{y}}\,\psi_n\,\frac{\partial \beta_j}{\partial \bar{y}} \\
&\qquad + \left(\frac{h}{b}\right)^2 \bar{c}_{kl}\,\bar{c}_{mn}\,\bar{c}_{op}\,\bar{c}_{ij}\,\phi_k\,\frac{\partial \phi_m}{\partial \bar{x}}\,\frac{\partial \phi_o}{\partial \bar{x}}\,\phi_i\,\frac{\partial \psi_l}{\partial \bar{y}}\,\psi_n \psi_p\,\frac{\partial \psi_j}{\partial \bar{y}}
\end{aligned}
$$

$$+2\frac{h}{b}\bar{c}_{kl}\bar{b}_{ij}\frac{\partial\bar{z}_p}{\partial\bar{x}}\phi_k\frac{\partial\alpha_i}{\partial\bar{x}}\frac{\partial\psi_l}{\partial\bar{y}}\beta_j+2\frac{a}{b}\frac{h}{b}\bar{c}_{kl}\bar{a}_{ij}\frac{\partial\bar{z}_p}{\partial\bar{x}}\phi_k\alpha_i\frac{\partial\psi_l}{\partial\bar{y}}\frac{\partial\beta_j}{\partial\bar{y}}$$

$$+2\frac{h}{b}\bar{c}_{kl}\bar{b}_{ij}\frac{\partial\bar{z}_p}{\partial\bar{y}}\frac{\partial\phi_k}{\partial\bar{x}}\frac{\partial\alpha_i}{\partial\bar{x}}\psi_l\beta_j+2\frac{a}{b}\frac{h}{b}\bar{c}_{kl}\bar{a}_{ij}\frac{\partial\bar{z}_p}{\partial\bar{y}}\frac{\partial\phi_k}{\partial\bar{x}}\alpha_i\psi_l\frac{\partial\beta_j}{\partial\bar{y}}$$

$$+2\left(\frac{h}{b}\right)^2\bar{c}_{kl}\bar{c}_{mn}\bar{c}_{ij}\frac{\partial\bar{z}_p}{\partial\bar{x}}\phi_k\frac{\partial\phi_m}{\partial\bar{x}}\phi_i\frac{\partial\psi_l}{\partial\bar{y}}\psi_n\frac{\partial\psi_j}{\partial\bar{y}}$$

$$+2\left(\frac{h}{b}\right)^2\bar{c}_{kl}\bar{c}_{mn}\bar{c}_{ij}\frac{\partial\bar{z}_p}{\partial\bar{y}}\frac{\partial\phi_k}{\partial\bar{x}}\phi_m\frac{\partial\phi_i}{\partial\bar{x}}\psi_l\frac{\partial\psi_n}{\partial\bar{y}}\psi_j$$

$$+2\left(\frac{h}{b}\right)^2\bar{c}_{kl}\bar{c}_{ij}\frac{\partial\bar{z}_p}{\partial\bar{x}}\frac{\partial\bar{z}_p}{\partial\bar{y}}\frac{\partial\phi_k}{\partial\bar{x}}\phi_i\psi_l\frac{\partial\psi_j}{\partial\bar{y}}$$

$$+\left(\frac{h}{b}\right)^2\bar{c}_{kl}\bar{c}_{ij}\left(\frac{\partial\bar{z}_p}{\partial\bar{x}}\right)^2\phi_k\phi_i\frac{\partial\psi_l}{\partial\bar{y}}\frac{\partial\psi_j}{\partial\bar{y}}$$

$$+\left(\frac{h}{b}\right)^2\bar{c}_{kl}\bar{c}_{ij}\left(\frac{\partial\bar{z}_p}{\partial\bar{y}}\right)^2\frac{\partial\phi_k}{\partial\bar{x}}\frac{\partial\phi_i}{\partial\bar{x}}\psi_l\psi_j\Bigg]$$

$$-\left(1+E^*\nu_{xy}\frac{\alpha_y}{\alpha_x}\right)\Bigg[\frac{a}{h}\bar{a}_{ij}\frac{\partial\alpha_i}{\partial\bar{x}}\beta_j+\frac{1}{2}\bar{c}_{kl}\bar{c}_{ij}\frac{\partial\phi_k}{\partial\bar{x}}\frac{\partial\phi_i}{\partial\bar{x}}\psi_l\psi_j$$

$$+\bar{c}_{ij}\frac{\partial\bar{z}_p}{\partial\bar{x}}\frac{\partial\phi_i}{\partial\bar{x}}\psi_j\Bigg]\alpha_x\,\Delta T(\bar{x},\bar{y})$$

$$-E^*\left(\frac{\alpha_y}{\alpha_x}+\nu_{xy}\right)\frac{a}{b}\Bigg[\frac{a}{h}\bar{b}_{ij}\alpha_i\frac{\partial\beta_j}{\partial\bar{y}}+\frac{1}{2}\frac{a}{b}\bar{c}_{kl}\bar{c}_{ij}\phi_k\phi_i\frac{\partial\psi_l}{\partial\bar{y}}\frac{\partial\psi_j}{\partial\bar{y}}$$

$$+\frac{a}{b}\bar{c}_{ij}\frac{\partial\bar{z}_p}{\partial\bar{y}}\phi_i\frac{\partial\psi_j}{\partial\bar{y}}\Bigg]\alpha_x\Delta T(\bar{x},\bar{y})$$

$$+\frac{(1-E^*\nu_{xy}^2)}{E_xh}\left[\frac{a}{h}\bar{a}_{ij}\frac{\partial\alpha_i}{\partial\bar{x}}\beta_j+\frac{1}{2}\bar{c}_{kl}\bar{c}_{ij}\frac{\partial\phi_k}{\partial\bar{x}}\frac{\partial\phi_i}{\partial\bar{x}}\psi_l\psi_j+\bar{c}_{ij}\frac{\partial\bar{z}_p}{\partial\bar{x}}\frac{\partial\phi_i}{\partial\bar{x}}\psi_j\right]N_x^a$$

$$+\frac{(1-E^*\nu_{xy}^2)}{E_xh}\frac{a}{b}\Bigg[\frac{a}{h}\bar{b}_{ij}\alpha_i\frac{\partial\beta_j}{\partial\bar{y}}+\frac{1}{2}\frac{a}{b}\bar{c}_{kl}\bar{c}_{ij}\phi_k\phi_i\frac{\partial\psi_l}{\partial\bar{y}}\frac{\partial\psi_j}{\partial\bar{y}}$$

$$+\frac{a}{b}\bar{c}_{ij}\frac{\partial\bar{z}_p}{\partial\bar{y}}\phi_i\frac{\partial\psi_j}{\partial\bar{y}}\Bigg]N_y^a\Bigg\}d\bar{x}\,d\bar{y} \qquad (29)$$

One other assumption must be made before the three nondimensional equations of motion can be derived from the above equations. The only forces acting on the plate that have not been explicitly defined to this point are the nonconservative forces. It is assumed that all the nonconservative forces act in the z direction only. Thus, the in-plane nonconservative forces are neglected.

The equation of motion in each direction is determined by substituting each generalized

coordinate into the nondimensional form of Lagrange's equation, (22),

$$x: \qquad \frac{\partial U_S}{\partial \bar{a}_{ij}} = 0 \tag{30.1}$$

$$y: \qquad \frac{\partial U_S}{\partial \bar{b}_{kl}} = 0 \tag{30.2}$$

$$z: \qquad \frac{d}{d\tau}\left(\frac{\partial T}{\partial \dot{\bar{c}}_{mn}}\right) + \frac{\partial U_B}{\partial \bar{c}_{mn}} + \frac{\partial U_S}{\partial \bar{c}_{mn}} + \bar{Q}_{zmn} = 0 \tag{30.3}$$

<u>x Equation</u>

The x equation is formed by substituting the stretching strain energy expression into Equation (30.1). Note the kinetic and bending strain energy expressions do not have any terms containing $\bar{a}_{ij}$. The derivative of U_S with respect to $\bar{a}_{ij}$ is

$$\begin{aligned}
\iint \Bigg\{ & 2\bar{a}_{kl}\frac{\partial \alpha_k}{\partial \bar{x}}\frac{\partial \alpha_i}{\partial \bar{x}}\beta_l\beta_j + \frac{h}{a}\bar{c}_{kl}\bar{c}_{mn}\frac{\partial \phi_k}{\partial \bar{x}}\frac{\partial \phi_m}{\partial \bar{x}}\frac{\partial \alpha_i}{\partial \bar{x}}\psi_l\psi_n\beta_j + 2\frac{h}{a}\bar{c}_{kl}\frac{\partial \bar{z}_p}{\partial \bar{x}}\frac{\partial \phi_k}{\partial \bar{x}}\frac{\partial \alpha_i}{\partial \bar{x}}\psi_l\beta_j \\
& + 2E^*\nu_{xy}\frac{a}{b}\Bigg[\bar{b}_{kl}\alpha_k\frac{\partial \alpha_i}{\partial \bar{x}}\frac{\partial \beta_l}{\partial \bar{y}}\beta_j + \frac{1}{2}\frac{h}{b}\bar{c}_{kl}\bar{c}_{mn}\phi_k\phi_m\frac{\partial \alpha_i}{\partial \bar{x}}\frac{\partial \psi_l}{\partial \bar{y}}\frac{\partial \psi_n}{\partial \bar{y}}\beta_j \\
& \qquad + \frac{h}{b}\bar{c}_{kl}\frac{\partial \bar{z}_p}{\partial \bar{y}}\phi_k\frac{\partial \alpha_i}{\partial \bar{x}}\frac{\partial \psi_l}{\partial \bar{y}}\beta_j\Bigg] \\
& + 2D_{xy}\frac{a}{b}\Bigg[\frac{a}{b}\bar{a}_{kl}\alpha_k\alpha_i\frac{\partial \beta_l}{\partial \bar{y}}\frac{\partial \beta_j}{\partial \bar{y}} + \bar{b}_{kl}\frac{\partial \alpha_k}{\partial \bar{x}}\alpha_i\beta_l\frac{\partial \beta_j}{\partial \bar{y}} + \frac{h}{b}\bar{c}_{kl}\bar{c}_{mn}\phi_k\frac{\partial \phi_m}{\partial \bar{x}}\alpha_i\frac{\partial \psi_l}{\partial \bar{y}}\psi_n\frac{\partial \beta_j}{\partial \bar{y}} \\
& \qquad + \frac{h}{b}\bar{c}_{kl}\frac{\partial \bar{z}_p}{\partial \bar{x}}\phi_k\alpha_i\frac{\partial \psi_l}{\partial \bar{y}}\frac{\partial \beta_j}{\partial \bar{y}} + \frac{h}{b}\bar{c}_{kl}\frac{\partial \bar{z}_p}{\partial \bar{y}}\frac{\partial \phi_k}{\partial \bar{x}}\alpha_i\psi_l\frac{\partial \beta_j}{\partial \bar{y}}\Bigg] \\
& - \left(1 + E^*\nu_{xy}\frac{\alpha_y}{\alpha_x}\right)\frac{a}{h}\alpha_x\Delta T(\bar{x},\bar{y})\frac{\partial \alpha_i}{\partial \bar{x}}\beta_j + \frac{(1 - E^*\nu_{xy}^2)}{E_x h}\frac{a}{h}N_x^a\frac{\partial \alpha_i}{\partial \bar{x}}\beta_j\Bigg\}\,d\bar{x}\,d\bar{y} = 0
\end{aligned}$$

Simplifying for a specific ij and regrouping yields

$$\begin{aligned}
& 2\left[\int \frac{\partial \alpha_k}{\partial \bar{x}}\frac{\partial \alpha_i}{\partial \bar{x}}d\bar{x}\int \beta_l\beta_j d\bar{y} + D_{xy}\left(\frac{a}{b}\right)^2\int \alpha_k\alpha_i d\bar{x}\int \frac{\partial \beta_l}{\partial \bar{y}}\frac{\partial \beta_j}{\partial \bar{y}}d\bar{y}\right]\bar{a}_{kl} \\
& + 2\frac{a}{b}\left[E^*\nu_{xy}\int \alpha_k\frac{\partial \alpha_i}{\partial \bar{x}}d\bar{x}\int \frac{\partial \beta_l}{\partial \bar{y}}\beta_j d\bar{y} + D_{xy}\int \frac{\partial \alpha_k}{\partial \bar{x}}\alpha_i d\bar{x}\int \beta_l\frac{\partial \beta_j}{\partial \bar{y}}d\bar{y}\right]\bar{b}_{kl} \\
& \qquad = -\frac{h}{a}\left[\int \frac{\partial \phi_k}{\partial \bar{x}}\frac{\partial \phi_m}{\partial \bar{x}}\frac{\partial \alpha_i}{\partial \bar{x}}d\bar{x}\int \psi_l\psi_n\beta_j d\bar{y}\right.
\end{aligned}$$

$$+ E^* \nu_{xy} \left(\frac{a}{b}\right)^2 \int \phi_k \phi_m \frac{\partial \alpha_i}{\partial \bar{x}} d\bar{x} \int \frac{\partial \psi_l}{\partial \bar{y}} \frac{\partial \psi_n}{\partial \bar{y}} \beta_j d\bar{y}$$

$$+ 2 D_{xy} \left(\frac{a}{b}\right)^2 \int \phi_k \frac{\partial \phi_m}{\partial \bar{x}} \alpha_i d\bar{x} \int \frac{\partial \psi_l}{\partial \bar{y}} \psi_n \frac{\partial \beta_j}{\partial \bar{y}} d\bar{y} \Bigg] \bar{c}_{kl} \bar{c}_{mn}$$

$$- 2 \frac{h}{a} \iint \left\{ \frac{\partial \bar{z}_p}{\partial \bar{x}} \left[\frac{\partial \phi_k}{\partial \bar{x}} \frac{\partial \alpha_i}{\partial \bar{x}} \psi_l \beta_j + D_{xy} \left(\frac{a}{b}\right)^2 \phi_k \alpha_i \frac{\partial \psi_l}{\partial \bar{y}} \frac{\partial \beta_j}{\partial \bar{y}} \right] \right.$$

$$\left. + \left(\frac{a}{b}\right)^2 \frac{\partial \bar{z}_p}{\partial \bar{y}} \left[E^* \nu_{xy} \phi_k \frac{\partial \alpha_i}{\partial \bar{x}} \frac{\partial \psi_l}{\partial \bar{y}} \beta_j + D_{xy} \frac{\partial \phi_k}{\partial \bar{x}} \alpha_i \psi_l \frac{\partial \beta_j}{\partial \bar{y}} \right] \right\} d\bar{x} d\bar{y} \bar{c}_{kl}$$

$$+ \alpha_x \left(1 + E^* \nu_{xy} \frac{\alpha_y}{\alpha_x}\right) \frac{a}{h} \iint \Delta T(\bar{x},\bar{y}) \frac{\partial \alpha_i}{\partial \bar{x}} \beta_j d\bar{x} d\bar{y}$$

$$- \frac{(1 - E^* \nu_{xy}^2)}{E_x h} \frac{a}{h} \int \frac{\partial \alpha_i}{\partial \bar{x}} d\bar{x} \int \beta_j d\bar{y} \; (N_x^a) \tag{31}$$

which is the nondimensional equation of motion in the x direction.

y Equation

The y equation is also formed by substituting only the stretching strain energy expression into Equation (30.2). The result is

$$\iint \left\{ 2E^* \left(\frac{a}{b}\right)^2 \left[\bar{b}_{kl} \alpha_k \alpha_i \frac{\partial \beta_l}{\partial \bar{y}} \frac{\partial \beta_j}{\partial \bar{y}} + \frac{1}{2} \frac{h}{b} \bar{c}_{kl} \bar{c}_{mn} \phi_k \phi_m \alpha_i \frac{\partial \psi_l}{\partial \bar{y}} \frac{\partial \psi_n}{\partial \bar{y}} \frac{\partial \beta_j}{\partial \bar{y}} \right. \right.$$

$$\left. + \frac{h}{b} \bar{c}_{kl} \frac{\partial \bar{z}_p}{\partial \bar{y}} \phi_k \alpha_i \frac{\partial \psi_l}{\partial \bar{y}} \frac{\partial \beta_j}{\partial \bar{y}} \right]$$

$$+ 2E^* \nu_{xy} \frac{a}{b} \left[\bar{a}_{kl} \frac{\partial \alpha_k}{\partial \bar{x}} \alpha_i \beta_l \frac{\partial \beta_j}{\partial \bar{y}} + \frac{1}{2} \frac{h}{a} \bar{c}_{kl} \bar{c}_{mn} \frac{\partial \phi_k}{\partial \bar{x}} \frac{\partial \phi_m}{\partial \bar{x}} \alpha_i \psi_l \psi_n \frac{\partial \beta_j}{\partial \bar{y}} \right.$$

$$\left. + \frac{h}{a} \bar{c}_{kl} \frac{\partial \bar{z}_p}{\partial \bar{x}} \frac{\partial \phi_k}{\partial \bar{x}} \alpha_i \psi_l \frac{\partial \beta_j}{\partial \bar{y}} \right]$$

$$+ 2 D_{xy} \left[\bar{b}_{kl} \frac{\partial \alpha_k}{\partial \bar{x}} \frac{\partial \alpha_i}{\partial \bar{x}} \beta_l \beta_j + \frac{a}{b} \bar{a}_{kl} \alpha_k \frac{\partial \alpha_i}{\partial \bar{x}} \frac{\partial \beta_l}{\partial \bar{y}} \beta_j \right.$$

$$+ \frac{h}{b} \bar{c}_{kl} \bar{c}_{mn} \frac{\partial \phi_k}{\partial \bar{x}} \phi_m \frac{\partial \alpha_i}{\partial \bar{x}} \psi_l \frac{\partial \psi_n}{\partial \bar{y}} \beta_j$$

$$\left. + \frac{h}{b} \bar{c}_{kl} \frac{\partial \bar{z}_p}{\partial \bar{x}} \phi_k \frac{\partial \alpha_i}{\partial \bar{x}} \frac{\partial \psi_l}{\partial \bar{y}} \beta_j + \frac{h}{b} \bar{c}_{kl} \frac{\partial \bar{z}_p}{\partial \bar{y}} \frac{\partial \phi_k}{\partial \bar{x}} \frac{\partial \alpha_i}{\partial \bar{x}} \psi_l \beta_j \right]$$

$$\left. - E^* \left(\frac{\alpha_y}{\alpha_x} + \nu_{xy}\right) \frac{a}{b} \frac{a}{h} \alpha_x \Delta T(\bar{x},\bar{y}) \alpha_i \frac{\partial \beta_j}{\partial \bar{y}} + \frac{(1 - E^* \nu_{xy}^2)}{E_x h} \frac{a}{b} \frac{a}{h} N_y^a \alpha_i \frac{\partial \beta_j}{\partial \bar{y}} \right\} d\bar{x} d\bar{y} = 0$$

Simplifying for a specific ij and regrouping yields

$$2\frac{a}{b}\left[E^{*}\nu_{xy}\int\frac{\partial\alpha_k}{\partial\bar{x}}\alpha_i\,d\bar{x}\int\beta_l\frac{\partial\beta_j}{\partial\bar{y}}\,d\bar{y}+D_{xy}\int\alpha_k\frac{\partial\alpha_i}{\partial\bar{x}}\,d\bar{x}\int\frac{\partial\beta_l}{\partial\bar{y}}\beta_j\,d\bar{y}\right]\bar{a}_{kl}$$

$$+2\left[E^{*}\left(\frac{a}{b}\right)^2\int\alpha_k\alpha_i\,d\bar{x}\int\frac{\partial\beta_l}{\partial\bar{y}}\frac{\partial\beta_j}{\partial\bar{y}}\,d\bar{y}+D_{xy}\int\frac{\partial\alpha_k}{\partial\bar{x}}\frac{\partial\alpha_i}{\partial\bar{x}}\,d\bar{x}\int\beta_l\beta_j\,d\bar{y}\right]\bar{b}_{kl}$$

$$=-\frac{h}{b}\left[E^{*}\left(\frac{a}{b}\right)^2\int\phi_k\phi_m\alpha_i\,d\bar{x}\int\frac{\partial\psi_l}{\partial\bar{y}}\frac{\partial\psi_n}{\partial\bar{y}}\frac{\partial\beta_j}{\partial\bar{y}}\,d\bar{y}\right.$$

$$+E^{*}\nu_{xy}\int\frac{\partial\phi_k}{\partial\bar{x}}\frac{\partial\phi_m}{\partial\bar{x}}\alpha_i\,d\bar{x}\int\psi_l\psi_n\frac{\partial\beta_j}{\partial\bar{y}}\,d\bar{y}$$

$$\left.+2D_{xy}\int\frac{\partial\phi_k}{\partial\bar{x}}\phi_m\frac{\partial\alpha_i}{\partial\bar{x}}\,d\bar{x}\int\psi_l\frac{\partial\psi_n}{\partial\bar{y}}\beta_j\,d\bar{y}\right]\bar{c}_{kl}\bar{c}_{mn}$$

$$-2\frac{h}{b}\int\!\!\int\left\{\frac{\partial\bar{z}_p}{\partial\bar{x}}\left[E^{*}\nu_{xy}\frac{\partial\phi_k}{\partial\bar{x}}\alpha_i\psi_l\frac{\partial\beta_j}{\partial\bar{y}}+D_{xy}\phi_k\frac{\partial\alpha_i}{\partial\bar{x}}\frac{\partial\psi_l}{\partial\bar{y}}\beta_j\right]\right.$$

$$\left.+\frac{\partial\bar{z}_p}{\partial\bar{y}}\left[E^{*}\left(\frac{a}{b}\right)^2\phi_k\alpha_i\frac{\partial\psi_l}{\partial\bar{y}}\frac{\partial\beta_j}{\partial\bar{y}}+D_{xy}\frac{\partial\phi_k}{\partial\bar{x}}\frac{\partial\alpha_i}{\partial\bar{x}}\psi_l\beta_j\right]\right\}d\bar{x}\,d\bar{y}\,\bar{c}_{kl}$$

$$+\alpha_x E^{*}\left(\frac{\alpha_y}{\alpha_x}+\nu_{xy}\right)\frac{a}{b}\frac{a}{h}\int\!\!\int\Delta T(\bar{x},\bar{y})\,\alpha_i\frac{\partial\beta_j}{\partial\bar{y}}\,d\bar{x}\,d\bar{y}$$

$$-\frac{(1-E^{*}\nu_{xy}^2)}{E_x h}\frac{a}{b}\frac{a}{h}\int\alpha_i\,d\bar{x}\int\frac{\partial\beta_j}{\partial\bar{y}}\,d\bar{y}\,(N_y^a)\qquad(32)$$

which is the nondimensional equation of motion in the y direction.

<u>z Equation</u>

The z equation is defined by substituting the kinetic, bending, and stretching energy expressions into (30.3). It is written here in terms of a specific ij.

$$\frac{1}{6}\left\{\int\phi_k\phi_i\,d\bar{x}\int\psi_l\psi_j\,d\bar{y}\right\}\ddot{\bar{c}}_{kl}$$

$$+\frac{1}{6}\left\{\int\frac{\partial^2\phi_k}{\partial\bar{x}^2}\frac{\partial^2\phi_i}{\partial\bar{x}^2}\,d\bar{x}\int\psi_l\psi_j\,d\bar{y}+E^{*}\left(\frac{a}{b}\right)^4\int\phi_k\phi_i\,d\bar{x}\int\frac{\partial^2\psi_l}{\partial\bar{y}^2}\frac{\partial^2\psi_j}{\partial\bar{y}^2}\,d\bar{y}\right.$$

$$+E^{*}\nu_{xy}\left(\frac{a}{b}\right)^2\left[\int\frac{\partial^2\phi_k}{\partial\bar{x}^2}\phi_i\,d\bar{x}\int\psi_l\frac{\partial^2\psi_j}{\partial\bar{y}^2}\,d\bar{y}+\int\phi_k\frac{\partial^2\phi_i}{\partial\bar{x}^2}\,d\bar{x}\int\frac{\partial^2\psi_l}{\partial\bar{y}^2}\psi_j\,d\bar{y}\right]$$

$$+4D_{xy}\left(\frac{a}{b}\right)^2\int\frac{\partial\phi_k}{\partial\bar{x}}\frac{\partial\phi_i}{\partial\bar{x}}\,d\bar{x}\int\frac{\partial\psi_l}{\partial\bar{y}}\frac{\partial\psi_j}{\partial\bar{y}}\,d\bar{y}$$

$$+12\iint\left[\left(\frac{\partial\bar{z}_p}{\partial\bar{x}}\right)^2\left(\frac{\partial\phi_k}{\partial\bar{x}}\frac{\partial\phi_i}{\partial\bar{x}}\psi_l\psi_j + D_{xy}\left(\frac{a}{b}\right)^2\phi_k\phi_i\frac{\partial\psi_l}{\partial\bar{y}}\frac{\partial\psi_j}{\partial\bar{y}}\right)\right.$$

$$+\left(\frac{a}{b}\right)^2\left(\frac{\partial\bar{z}_p}{\partial\bar{y}}\right)^2\left(E^*\left(\frac{a}{b}\right)^2\phi_k\phi_i\frac{\partial\psi_l}{\partial\bar{y}}\frac{\partial\psi_j}{\partial\bar{y}} + D_{xy}\frac{\partial\phi_k}{\partial\bar{x}}\frac{\partial\phi_i}{\partial\bar{x}}\psi_l\psi_j\right)$$

$$\left.+(E^*\nu_{xy}+D_{xy})\left(\frac{a}{b}\right)^2\frac{\partial\bar{z}_p}{\partial\bar{x}}\frac{\partial\bar{z}_p}{\partial\bar{y}}\left(\phi_k\frac{\partial\phi_i}{\partial\bar{x}}\frac{\partial\psi_l}{\partial\bar{y}}\psi_j + \frac{\partial\phi_k}{\partial\bar{x}}\phi_i\psi_l\frac{\partial\psi_j}{\partial\bar{y}}\right)\right]d\bar{x}\,d\bar{y}\Bigg\}\,\bar{c}_{kl}$$

$$+\iint\left\{\frac{\partial\bar{z}_p}{\partial\bar{x}}\left[3\frac{\partial\phi_k}{\partial\bar{x}}\frac{\partial\phi_m}{\partial\bar{x}}\frac{\partial\phi_i}{\partial\bar{x}}\psi_l\psi_n\psi_j + (E^*\nu_{xy}+2D_{xy})\left(\frac{a}{b}\right)^2\phi_k\phi_m\frac{\partial\phi_i}{\partial\bar{x}}\frac{\partial\psi_l}{\partial\bar{y}}\frac{\partial\psi_n}{\partial\bar{y}}\psi_j\right.\right.$$

$$\left.+2(E^*\nu_{xy}+2D_{xy})\left(\frac{a}{b}\right)^2\phi_k\frac{\partial\phi_m}{\partial\bar{x}}\phi_i\frac{\partial\psi_l}{\partial\bar{y}}\psi_n\frac{\partial\psi_j}{\partial\bar{y}}\right]$$

$$+\left(\frac{a}{b}\right)^2\frac{\partial\bar{z}_p}{\partial\bar{y}}\left[3E^*\left(\frac{a}{h}\right)^2\phi_k\phi_m\phi_i\frac{\partial\psi_l}{\partial\bar{y}}\frac{\partial\psi_n}{\partial\bar{y}}\frac{\partial\psi_j}{\partial\bar{y}}\right.$$

$$+2(E^*\nu_{xy}+2D_{xy})\frac{\partial\phi_k}{\partial\bar{x}}\phi_m\frac{\partial\phi_i}{\partial\bar{x}}\psi_l\frac{\partial\psi_n}{\partial\bar{y}}\psi_j$$

$$\left.\left.+(E^*\nu_{xy}+2D_{xy})\frac{\partial\phi_k}{\partial\bar{x}}\frac{\partial\phi_m}{\partial\bar{x}}\phi_i\psi_l\psi_n\frac{\partial\psi_j}{\partial\bar{y}}\right]\right\}d\bar{x}\,d\bar{y}\quad\bar{c}_{kl}\bar{c}_{mn}$$

$$+\left\{\int\frac{\partial\phi_k}{\partial\bar{x}}\frac{\partial\phi_m}{\partial\bar{x}}\frac{\partial\phi_o}{\partial\bar{x}}\frac{\partial\phi_i}{\partial\bar{x}}d\bar{x}\int\psi_l\psi_n\psi_p\psi_j\,d\bar{y}\right.$$

$$+E^*\left(\frac{a}{b}\right)^4\int\phi_k\phi_m\phi_o\phi_i\,d\bar{x}\int\frac{\partial\psi_l}{\partial\bar{y}}\frac{\partial\psi_n}{\partial\bar{y}}\frac{\partial\psi_p}{\partial\bar{y}}\frac{\partial\psi_j}{\partial\bar{y}}d\bar{y}$$

$$+(E^*\nu_{xy}+2D_{xy})\left(\frac{a}{b}\right)^2\int\phi_k\frac{\partial\phi_m}{\partial\bar{x}}\frac{\partial\phi_o}{\partial\bar{x}}\phi_i\,d\bar{x}\int\frac{\partial\psi_l}{\partial\bar{y}}\psi_n\psi_p\frac{\partial\psi_j}{\partial\bar{y}}d\bar{y}$$

$$\left.+(E^*\nu_{xy}+2D_{xy})\left(\frac{a}{b}\right)^2\int\frac{\partial\phi_k}{\partial\bar{x}}\phi_m\phi_o\frac{\partial\phi_i}{\partial\bar{x}}d\bar{x}\int\psi_l\frac{\partial\psi_n}{\partial\bar{y}}\frac{\partial\psi_p}{\partial\bar{y}}\psi_j\,d\bar{y}\right\}\bar{c}_{kl}\bar{c}_{mn}\bar{c}_{op}$$

$$+2\frac{a}{h}\left\{\left[\int\frac{\partial\alpha_k}{\partial\bar{x}}\frac{\partial\phi_m}{\partial\bar{x}}\frac{\partial\phi_i}{\partial\bar{x}}d\bar{x}\int\beta_l\psi_n\psi_j\,d\bar{y} + E^*\nu_{xy}\left(\frac{a}{b}\right)^2\int\frac{\partial\alpha_k}{\partial\bar{x}}\phi_m\phi_i\,d\bar{x}\int\beta_l\frac{\partial\psi_n}{\partial\bar{y}}\frac{\partial\psi_j}{\partial\bar{y}}d\bar{y}\right.\right.$$

$$+D_{xy}\left(\frac{a}{b}\right)^2\int\alpha_k\phi_m\frac{\partial\phi_i}{\partial\bar{x}}d\bar{x}\int\frac{\partial\beta_l}{\partial\bar{y}}\frac{\partial\psi_n}{\partial\bar{y}}\psi_j\,d\bar{y}$$

$$\left.+D_{xy}\left(\frac{a}{b}\right)^2\int\alpha_k\frac{\partial\phi_m}{\partial\bar{x}}\phi_i\,d\bar{x}\int\frac{\partial\beta_l}{\partial\bar{y}}\psi_n\frac{\partial\psi_j}{\partial\bar{y}}d\bar{y}\right]\bar{c}_{mn}$$

$$+\iint\left[\frac{\partial\bar{z}_p}{\partial\bar{x}}\left(\frac{\partial\alpha_k}{\partial\bar{x}}\frac{\partial\phi_i}{\partial\bar{x}}\beta_l\psi_j + D_{xy}\left(\frac{a}{b}\right)^2\alpha_k\phi_i\frac{\partial\beta_l}{\partial\bar{y}}\frac{\partial\psi_j}{\partial\bar{y}}\right)\right.$$

$$\left.\left.+\left(\frac{a}{b}\right)^2\frac{\partial\bar{z}_p}{\partial\bar{y}}\left(E^*\nu_{xy}\frac{\partial\alpha_k}{\partial\bar{x}}\phi_i\beta_l\frac{\partial\psi_j}{\partial\bar{y}} + D_{xy}\alpha_k\frac{\partial\phi_i}{\partial\bar{x}}\frac{\partial\beta_l}{\partial\bar{y}}\psi_j\right)\right]d\bar{x}\,d\bar{y}\right\}\bar{a}_{kl}$$

$$
\begin{aligned}
&+2\frac{a}{b}\frac{a}{h}\Bigg\{\Bigg[E^*\left(\frac{a}{b}\right)^2\int\alpha_k\phi_m\phi_i\,d\bar{x}\int\frac{\partial\beta_l}{\partial\bar{y}}\frac{\partial\psi_n}{\partial\bar{y}}\frac{\partial\psi_j}{\partial\bar{y}}\,d\bar{y}\\
&\qquad+E^*\nu_{xy}\int\alpha_k\frac{\partial\phi_m}{\partial\bar{x}}\frac{\partial\phi_i}{\partial\bar{x}}\,d\bar{x}\int\frac{\partial\beta_l}{\partial\bar{y}}\psi_n\psi_j\,d\bar{y}\\
&\qquad+D_{xy}\int\frac{\partial\alpha_k}{\partial\bar{x}}\phi_m\frac{\partial\phi_i}{\partial\bar{x}}\,d\bar{x}\int\beta_l\frac{\partial\psi_n}{\partial\bar{y}}\psi_j\,d\bar{y}\\
&\qquad+D_{xy}\int\frac{\partial\alpha_k}{\partial\bar{x}}\frac{\partial\phi_m}{\partial\bar{x}}\phi_i\,d\bar{x}\int\beta_l\psi_n\frac{\partial\psi_j}{\partial\bar{y}}\,d\bar{y}\Bigg]\bar{c}_{mn}\\
&\qquad+\iint\Bigg[\frac{\partial\bar{z}_p}{\partial\bar{x}}\left(E^*\nu_{xy}\alpha_k\frac{\partial\phi_i}{\partial\bar{x}}\frac{\partial\beta_l}{\partial\bar{y}}\psi_j+D_{xy}\frac{\partial\alpha_k}{\partial\bar{x}}\phi_i\beta_l\frac{\partial\psi_j}{\partial\bar{y}}\right)\\
&\qquad\qquad+\frac{\partial\bar{z}_p}{\partial\bar{y}}\left(E^*\left(\frac{a}{b}\right)^2\alpha_k\phi_i\frac{\partial\beta_l}{\partial\bar{y}}\frac{\partial\psi_j}{\partial\bar{y}}+D_{xy}\frac{\partial\alpha_i}{\partial\bar{x}}\frac{\partial\phi_i}{\partial\bar{x}}\beta_l\psi_j\right)\Bigg]d\bar{x}\,d\bar{y}\Bigg\}\bar{b}_{kl}\\
&-\alpha_x\left(1+E^*\nu_{xy}\frac{\alpha_y}{\alpha_x}\right)\left(\frac{a}{h}\right)^2\Bigg\{\iint\Delta T(\bar{x},\bar{y})\left[\frac{\partial\phi_k}{\partial\bar{x}}\frac{\partial\phi_i}{\partial\bar{x}}\psi_l\psi_j\,d\bar{x}\,d\bar{y}\right]\bar{c}_{kl}\\
&\qquad\qquad+\iint\Delta T(\bar{x},\bar{y})\frac{\partial\bar{z}_p}{\partial\bar{x}}\frac{\partial\phi_i}{\partial\bar{x}}\psi_j\,d\bar{x}\,d\bar{y}\Bigg\}\\
&-\alpha_x E^*\left(\frac{\alpha_y}{\alpha_x}+\nu_{xy}\right)\left(\frac{a}{b}\right)^2\left(\frac{a}{h}\right)^2\Bigg\{\iint\Delta T(\bar{x},\bar{y})\left[\phi_k\phi_i\frac{\partial\psi_l}{\partial\bar{y}}\frac{\partial\psi_j}{\partial\bar{y}}\,d\bar{x}\,d\bar{y}\right]\bar{c}_{kl}\\
&\qquad\qquad+\iint\Delta T(\bar{x},\bar{y})\frac{\partial\bar{z}_p}{\partial\bar{y}}\phi_i\frac{\partial\psi_j}{\partial\bar{y}}\,d\bar{x}\,d\bar{y}\Bigg\}\\
&+\frac{(1-E^*\nu_{xy}^2)}{E_x h}\left(\frac{a}{h}\right)^2\Bigg\{\left[\int\frac{\partial\phi_k}{\partial\bar{x}}\frac{\partial\phi_i}{\partial\bar{x}}\,d\bar{x}\int\psi_l\psi_j\,d\bar{y}\right]\bar{c}_{kl}+\iint\frac{\partial\bar{z}_p}{\partial\bar{x}}\frac{\partial\phi_i}{\partial\bar{x}}\psi_j\,d\bar{x}\,d\bar{y}\Bigg\}\left(N_x^a\right)\\
&+\frac{(1-E^*\nu_{xy}^2)}{E_x h}\left(\frac{a}{b}\right)^2\left(\frac{a}{h}\right)^2\Bigg\{\left[\int\phi_k\phi_i\,d\bar{x}\int\frac{\partial\psi_l}{\partial\bar{y}}\frac{\partial\psi_j}{\partial\bar{y}}\,d\bar{y}\right]\bar{c}_{kl}+\iint\frac{\partial\bar{z}_p}{\partial\bar{y}}\phi_i\frac{\partial\psi_j}{\partial\bar{y}}\,d\bar{x}\,d\bar{y}\Bigg\}\left(N_y^a\right)\\
&\qquad\qquad+\frac{1}{6D_x}\frac{a}{b}\left(\frac{a}{h}\right)^2\bar{Q}_{zij}=0
\end{aligned}
\tag{33}
$$

SECTION IV

Isotropic Panels

The nondimensional equations of motion for a panel made of an isotropic material rather than an orthotropic material assumed previously are stated here with little derivation. The stress-strain relations for a thin isotropic panel under an applied temperature differential and in-plane loads are

$$\sigma_x = \frac{E}{(1-\nu^2)}\left[\epsilon_x + \nu\epsilon_y - (1+\nu)\alpha\,\Delta T\right] + \frac{1}{h}N_x^{\,a} \tag{34.1}$$

$$\sigma_x = \frac{E}{(1-\nu^2)}\left[\epsilon_y + \nu\epsilon_x - (1+\nu)\alpha\,\Delta T\right] + \frac{1}{h}N_y^{\,a} \tag{34.2}$$

$$\tau_{xy} = G\gamma_{xy} \tag{34.3}$$

Substituting Equations (34) into the total linear strain energy expression, (13), yields

$$U = \frac{E}{2(1-\nu^2)}\iiint\left[\epsilon_x^2 + \epsilon_y^2 + 2\nu\epsilon_x\epsilon_y + \frac{(1-\nu)}{2}\gamma_{xy}^2 - (1+\nu)\alpha\,\Delta T(x,y)(\epsilon_x+\epsilon_y)\right.$$
$$\left. + \frac{(1-\nu^2)}{Eh}\left(N_x^a\epsilon_x + N_y^a\epsilon_y\right)\right]dx\,dy\,dz \tag{35}$$

and substituting the strain-displacement relations, (16), into (35) gives

$$U = \frac{E}{2(1-\nu^2)}\iiint\left\{\epsilon_{xo}^2 + \epsilon_{yo}^2 + 2\nu\epsilon_{xo}\epsilon_{yo} + \frac{(1-\nu)}{2}\gamma_{xyo}^2 - (1+\nu)\alpha\,\Delta T(x,y)(\epsilon_{xo}+\epsilon_{yo})\right.$$
$$+ \frac{(1-\nu^2)}{Eh}\left(N_x^a\epsilon_{xo} + N_y^a\epsilon_{yo}\right)$$
$$+ z^2\left[\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + \left(\frac{\partial^2 w}{\partial y^2}\right)^2 + 2\nu\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} + 2(1-\nu)\left(\frac{\partial^2 w}{\partial x\,\partial y}\right)^2\right]$$
$$- 2z\left[\epsilon_{xo}\frac{\partial^2 w}{\partial x^2} + \epsilon_{yo}\frac{\partial^2 w}{\partial y^2} + \nu\left(\epsilon_{xo}\frac{\partial^2 w}{\partial y^2} + \epsilon_{yo}\frac{\partial^2 w}{\partial x^2}\right)\right.$$
$$+ (1-\nu)\gamma_{xyo}\frac{\partial^2 w}{\partial x\partial y} + \frac{(1+\nu)}{2}\alpha\,\Delta T(x,y)\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right)$$
$$\left.\left. + \frac{(1-\nu^2)}{2Eh}\left(N_x^a\frac{\partial^2 w}{\partial x^2} + N_y^a\frac{\partial^2 w}{\partial y^2}\right)\right]\right\}dx\,dy\,dz$$

Integrating with respect to z from $-\frac{h}{2}$ to $\frac{h}{2}$ and separating the result into the bending and stretching equations using (9) produces

$$U_B = \frac{D}{2}\iint\left[\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + \left(\frac{\partial^2 w}{\partial y^2}\right)^2 + 2\nu\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} + 2(1-\nu)\left(\frac{\partial^2 w}{\partial x\,\partial y}\right)^2\right]dx\,dy \tag{36}$$

and

$$\begin{aligned} U_S = \frac{Eh}{2(1-\nu^2)}\iint\Bigg\{&\left[\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 + \frac{\partial z_p}{\partial x}\frac{\partial w}{\partial x}\right]^2 + \left[\frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 + \frac{\partial z_p}{\partial y}\frac{\partial w}{\partial y}\right]^2 \\ &+ 2\nu\left[\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 + \frac{\partial z_p}{\partial x}\frac{\partial w}{\partial x}\right]\left[\frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 + \frac{\partial z_p}{\partial y}\frac{\partial w}{\partial y}\right] \\ &+ \frac{(1-\nu)}{2}\left[\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y} + \frac{\partial z_p}{\partial x}\frac{\partial w}{\partial y} + \frac{\partial z_p}{\partial y}\frac{\partial w}{\partial x}\right]^2 \\ &- \alpha(1+\nu)\left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 + \frac{\partial z_p}{\partial x}\frac{\partial w}{\partial x} + \frac{\partial z_p}{\partial y}\frac{\partial w}{\partial y}\right]\Delta T(x,y) \\ &+ \frac{(1-\nu^2)}{Eh}\left[\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 + \frac{\partial z_p}{\partial x}\frac{\partial w}{\partial x}\right]\left(N_x^a\right) \\ &+ \frac{(1-\nu^2)}{Eh}\left[\frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 + \frac{\partial z_p}{\partial y}\frac{\partial w}{\partial y}\right]\left(N_y^a\right)\Bigg\}dx\,dy \end{aligned} \tag{37}$$

Equations (12), (36), and (37) are nondimensionalized using the parameters defined in (21) but replacing D_x with D. The nondimensional kinetic energy is

$$T = \frac{D}{2}\frac{b}{a}\left(\frac{h}{a}\right)^2\iint\left(\frac{\partial\bar{w}}{\partial\tau}\right)^2 d\bar{x}\,d\bar{y} \tag{38}$$

the bending strain energy is

$$\begin{aligned} U_B = \frac{D}{2}\frac{b}{a}\left(\frac{h}{a}\right)^2\iint\Bigg[&\left(\frac{\partial^2\bar{w}}{\partial\bar{x}^2}\right)^2 + \left(\frac{a}{b}\right)^4\left(\frac{\partial^2\bar{w}}{\partial\bar{y}^2}\right)^2 + 2\nu\left(\frac{a}{b}\right)^2\frac{\partial^2\bar{w}}{\partial\bar{x}^2}\frac{\partial^2\bar{w}}{\partial\bar{y}^2} \\ &+ 2(1-\nu)\left(\frac{a}{b}\right)^2\left(\frac{\partial^2\bar{w}}{\partial\bar{x}\partial\bar{y}}\right)^2\Bigg]d\bar{x}\,d\bar{y} \end{aligned} \tag{39}$$

and the stretching strain energy is

$$\begin{aligned} U_S = 6D\frac{b}{a}\iint\Bigg\{&\left[\frac{\partial\bar{u}}{\partial\bar{x}} + \frac{1}{2}\frac{h}{a}\left(\frac{\partial\bar{w}}{\partial\bar{x}}\right)^2 + \frac{h}{a}\frac{\partial\bar{z}_p}{\partial\bar{x}}\frac{\partial\bar{w}}{\partial\bar{x}}\right]^2 + \left(\frac{a}{b}\right)^2\left[\frac{\partial\bar{v}}{\partial\bar{y}} + \frac{1}{2}\frac{h}{b}\left(\frac{\partial\bar{w}}{\partial\bar{y}}\right)^2 + \frac{h}{b}\frac{\partial\bar{z}_p}{\partial\bar{y}}\frac{\partial\bar{w}}{\partial\bar{y}}\right]^2 \\ &+ 2\nu\frac{a}{b}\left[\frac{\partial\bar{u}}{\partial\bar{x}} + \frac{1}{2}\frac{h}{a}\left(\frac{\partial\bar{w}}{\partial\bar{x}}\right)^2 + \frac{h}{a}\frac{\partial\bar{z}_p}{\partial\bar{x}}\frac{\partial\bar{w}}{\partial\bar{x}}\right]\left[\frac{\partial\bar{v}}{\partial\bar{y}} + \frac{1}{2}\frac{h}{b}\left(\frac{\partial\bar{w}}{\partial\bar{y}}\right)^2 + \frac{h}{b}\frac{\partial\bar{z}_p}{\partial\bar{y}}\frac{\partial\bar{w}}{\partial\bar{y}}\right] \\ &+ \frac{(1-\nu)}{2}\left[\frac{\partial\bar{v}}{\partial\bar{x}} + \frac{a}{b}\frac{\partial\bar{u}}{\partial\bar{y}} + \frac{h}{b}\frac{\partial\bar{w}}{\partial\bar{x}}\frac{\partial\bar{w}}{\partial\bar{y}} + \frac{h}{b}\frac{\partial\bar{z}_p}{\partial\bar{x}}\frac{\partial\bar{w}}{\partial\bar{y}} + \frac{h}{b}\frac{\partial\bar{z}_p}{\partial\bar{y}}\frac{\partial\bar{w}}{\partial\bar{x}}\right]^2 \\ &- \alpha(1+\nu)\left[\frac{a}{h}\frac{\partial\bar{u}}{\partial\bar{x}} + \frac{a}{b}\frac{a}{h}\frac{\partial\bar{v}}{\partial\bar{y}} + \frac{1}{2}\left(\frac{\partial\bar{w}}{\partial\bar{x}}\right)^2 + \frac{1}{2}\left(\frac{a}{b}\right)^2\left(\frac{\partial\bar{w}}{\partial\bar{y}}\right)^2\right. \end{aligned}$$

$$+ \frac{\partial \bar{z}_p}{\partial \bar{x}} \frac{\partial \bar{w}}{\partial \bar{x}} + \left(\frac{a}{b}\right)^2 \frac{\partial \bar{z}_p}{\partial \bar{y}} \frac{\partial \bar{w}}{\partial \bar{y}} \Bigg] \Delta T(x,y)$$

$$+ \frac{(1-\nu^2)}{Eh} \left[\frac{a}{h} \frac{\partial \bar{u}}{\partial \bar{x}} + \frac{1}{2} \left(\frac{\partial \bar{w}}{\partial \bar{x}} \right)^2 + \frac{\partial \bar{z}_p}{\partial \bar{x}} \frac{\partial \bar{w}}{\partial \bar{x}} \right] \left(N_x^a \right)$$

$$+ \frac{(1-\nu^2)}{Eh} \frac{a}{b} \left[\frac{a}{h} \frac{\partial \bar{v}}{\partial \bar{y}} + \frac{1}{2} \frac{a}{b} \left(\frac{\partial \bar{w}}{\partial \bar{y}} \right)^2 + \frac{a}{b} \frac{\partial \bar{z}_p}{\partial \bar{y}} \frac{\partial \bar{w}}{\partial \bar{y}} \right] \left(N_y^a \right) \Bigg\} d\bar{x}\, d\bar{y} \tag{40}$$

Isotropic x Equation

The nondimensional x equation simplified for a specific ij is

$$2 \left[\int \frac{\partial \alpha_k}{\partial \bar{x}} \frac{\partial \alpha_i}{\partial \bar{x}} d\bar{x} \int \beta_l \beta_j d\bar{y} + \frac{(1-\nu)}{2} \left(\frac{a}{b} \right)^2 \int \alpha_k \alpha_i d\bar{x} \int \frac{\partial \beta_l}{\partial \bar{y}} \frac{\partial \beta_j}{\partial \bar{y}} d\bar{y} \right] \bar{a}_{kl}$$

$$+ 2 \frac{a}{b} \left[\nu \int \alpha_k \frac{\partial \alpha_i}{\partial \bar{x}} d\bar{x} \int \frac{\partial \beta_l}{\partial \bar{y}} \beta_j d\bar{y} + \frac{(1-\nu)}{2} \int \frac{\partial \alpha_k}{\partial \bar{x}} \alpha_i d\bar{x} \int \beta_l \frac{\partial \beta_j}{\partial \bar{y}} d\bar{y} \right] \bar{b}_{kl}$$

$$= - \frac{h}{a} \Bigg[\int \frac{\partial \phi_k}{\partial \bar{x}} \frac{\partial \phi_m}{\partial \bar{x}} \frac{\partial \alpha_i}{\partial \bar{x}} d\bar{x} \int \psi_l \psi_n \beta_j d\bar{y}$$

$$+ \nu \left(\frac{a}{b} \right)^2 \int \phi_k \phi_m \frac{\partial \alpha_i}{\partial \bar{x}} d\bar{x} \int \frac{\partial \psi_l}{\partial \bar{y}} \frac{\partial \psi_n}{\partial \bar{y}} \beta_j d\bar{y}$$

$$+ (1-\nu) \left(\frac{a}{b} \right)^2 \int \phi_k \frac{\partial \phi_m}{\partial \bar{x}} \alpha_i d\bar{x} \int \frac{\partial \psi_l}{\partial \bar{y}} \psi_n \frac{\partial \beta_j}{\partial \bar{y}} d\bar{y} \Bigg] \bar{c}_{kl} \bar{c}_{mn}$$

$$- 2 \frac{h}{a} \int \int \Bigg\{ \frac{\partial \bar{z}_p}{\partial \bar{x}} \left[\frac{\partial \phi_k}{\partial \bar{x}} \frac{\partial \alpha_i}{\partial \bar{x}} \psi_l \beta_j + \frac{(1-\nu)}{2} \left(\frac{a}{b} \right)^2 \phi_k \alpha_i \frac{\partial \psi_l}{\partial \bar{y}} \frac{\partial \beta_j}{\partial \bar{y}} \right]$$

$$+ \left(\frac{a}{b} \right)^2 \frac{\partial \bar{z}_p}{\partial \bar{y}} \left[\nu \phi_k \frac{\partial \alpha_i}{\partial \bar{x}} \frac{\partial \psi_l}{\partial \bar{y}} \beta_j + \frac{(1-\nu)}{2} \frac{\partial \phi_k}{\partial \bar{x}} \alpha_i \psi_l \frac{\partial \beta_j}{\partial \bar{y}} \right] \Bigg\} d\bar{x}\, d\bar{y}\, \bar{c}_{kl}$$

$$+ \alpha (1+\nu) \frac{a}{h} \int \int \Delta T(x,y) \frac{\partial \alpha_i}{\partial \bar{x}} \beta_j d\bar{x}\, d\bar{y} - \frac{(1-\nu^2)}{Eh} \frac{a}{h} \int \frac{\partial \alpha_i}{\partial \bar{x}} d\bar{x} \int \beta_j d\bar{y} \left(N_x^a \right) \tag{41}$$

Isotropic y Equation

The nondimensional y equation simplified for a specific ij is

$$2 \frac{a}{b} \left[\nu \int \frac{\partial \alpha_k}{\partial \bar{x}} \alpha_i d\bar{x} \int \beta_l \frac{\partial \beta_j}{\partial \bar{y}} d\bar{y} + \frac{(1-\nu)}{2} \int \alpha_k \frac{\partial \alpha_i}{\partial \bar{x}} d\bar{x} \int \frac{\partial \beta_l}{\partial \bar{y}} \beta_j d\bar{y} \right] \bar{a}_{kl}$$

$$+ 2 \left[\left(\frac{a}{b} \right)^2 \int \alpha_k \alpha_i d\bar{x} \int \frac{\partial \beta_l}{\partial \bar{y}} \frac{\partial \beta_j}{\partial \bar{y}} d\bar{y} + \frac{(1-\nu)}{2} \int \frac{\partial \alpha_k}{\partial \bar{x}} \frac{\partial \alpha_i}{\partial \bar{x}} d\bar{x} \int \beta_l \beta_j d\bar{y} \right] \bar{b}_{kl}$$

$$= -\frac{h}{b}\left[\left(\frac{a}{b}\right)^2\int \phi_k\phi_m\alpha_i\,d\bar{x}\int \frac{\partial\psi_l}{\partial\bar{y}}\frac{\partial\psi_n}{\partial\bar{y}}\frac{\partial\beta_j}{\partial\bar{y}}\,d\bar{y}\right.$$

$$+ \nu\int \frac{\partial\phi_k}{\partial\bar{x}}\frac{\partial\phi_m}{\partial\bar{x}}\alpha_i\,d\bar{x}\int \psi_l\psi_n\frac{\partial\beta_j}{\partial\bar{y}}\,d\bar{y}$$

$$\left. + (1-\nu)\int \frac{\partial\phi_k}{\partial\bar{x}}\phi_m\frac{\partial\alpha_i}{\partial\bar{x}}\,d\bar{x}\int \psi_l\frac{\partial\psi_n}{\partial\bar{y}}\beta_j\,d\bar{y}\right]\bar{c}_{kl}\bar{c}_{mn}$$

$$-2\frac{h}{b}\iint\left\{\frac{\partial\bar{z}_p}{\partial\bar{x}}\left[\nu\frac{\partial\phi_k}{\partial\bar{x}}\alpha_i\psi_l\frac{\partial\beta_j}{\partial\bar{y}} + \frac{(1-\nu)}{2}\phi_k\frac{\partial\alpha_i}{\partial\bar{x}}\frac{\partial\psi_l}{\partial\bar{y}}\beta_j\right]\right.$$

$$\left. + \frac{\partial\bar{z}_p}{\partial\bar{y}}\left[\left(\frac{a}{b}\right)^2\phi_k\alpha_i\frac{\partial\psi_l}{\partial\bar{y}}\frac{\partial\beta_j}{\partial\bar{y}} + \frac{(1-\nu)}{2}\frac{\partial\phi_k}{\partial\bar{x}}\frac{\partial\alpha_i}{\partial\bar{x}}\psi_l\beta_j\right]\right\}d\bar{x}\,d\bar{y}\,\bar{c}_{kl}$$

$$+ \alpha(1+\nu)\frac{a}{b}\frac{a}{h}\iint \Delta T(x,y)\,\alpha_i\frac{\partial\beta_j}{\partial\bar{y}}\,d\bar{x}\,d\bar{y}$$

$$-\frac{(1-\nu^2)}{Eh}\frac{a}{b}\frac{a}{h}\int \alpha_i\,d\bar{x}\int \frac{\partial\beta_j}{\partial\bar{y}}\,d\bar{y}\left(N_y^a\right) \tag{42}$$

Isotropic z Equation

The nondimensional z equation written in terms of a specific ij is

$$\frac{1}{6}\left\{\int \phi_k\phi_i\,d\bar{x}\int \psi_l\psi_j\,d\bar{y}\right\}\ddot{\bar{c}}_{kl}$$

$$+\frac{1}{6}\left\{\int \frac{\partial^2\phi_k}{\partial\bar{x}^2}\frac{\partial^2\phi_i}{\partial\bar{x}^2}\,d\bar{x}\int \psi_l\psi_j\,d\bar{y} + \left(\frac{a}{b}\right)^4\int \phi_k\phi_i\,d\bar{x}\int \frac{\partial^2\psi_l}{\partial\bar{y}^2}\frac{\partial^2\psi_j}{\partial\bar{y}^2}\,d\bar{y}\right.$$

$$+ \nu\left(\frac{a}{b}\right)^2\left[\int \frac{\partial^2\phi_k}{\partial\bar{x}^2}\phi_i\,d\bar{x}\int \psi_l\frac{\partial^2\psi_j}{\partial\bar{y}^2}\,d\bar{y} + \int \phi_k\frac{\partial^2\phi_i}{\partial\bar{x}^2}\,d\bar{x}\int \frac{\partial^2\psi_l}{\partial\bar{y}^2}\psi_j\,d\bar{y}\right]$$

$$+ 2(1-\nu)\left(\frac{a}{b}\right)^2\int \frac{\partial\phi_k}{\partial\bar{x}}\frac{\partial\phi_i}{\partial\bar{x}}\,d\bar{x}\int \frac{\partial\psi_l}{\partial\bar{y}}\frac{\partial\psi_j}{\partial\bar{y}}\,d\bar{y}$$

$$+ 12\iint\left[\left(\frac{\partial\bar{z}_p}{\partial\bar{x}}\right)^2\left(\frac{\partial\phi_k}{\partial\bar{x}}\frac{\partial\phi_i}{\partial\bar{x}}\psi_l\psi_j + \frac{(1-\nu)}{2}\left(\frac{a}{b}\right)^2\phi_k\phi_i\frac{\partial\psi_l}{\partial\bar{y}}\frac{\partial\psi_j}{\partial\bar{y}}\right)\right.$$

$$+\left(\frac{a}{b}\right)^2\left(\frac{\partial\bar{z}_p}{\partial\bar{y}}\right)^2\left(\left(\frac{a}{b}\right)^2\phi_k\phi_i\frac{\partial\psi_l}{\partial\bar{y}}\frac{\partial\psi_j}{\partial\bar{y}} + \frac{(1-\nu)}{2}\frac{\partial\phi_k}{\partial\bar{x}}\frac{\partial\phi_i}{\partial\bar{x}}\psi_l\psi_j\right)$$

$$\left.\left. + \frac{(1+\nu)}{2}\left(\frac{a}{b}\right)^2\frac{\partial\bar{z}_p}{\partial\bar{x}}\frac{\partial\bar{z}_p}{\partial\bar{y}}\left(\phi_k\frac{\partial\phi_i}{\partial\bar{x}}\frac{\partial\psi_l}{\partial\bar{y}}\psi_j + \frac{\partial\phi_k}{\partial\bar{x}}\phi_i\psi_l\frac{\partial\psi_j}{\partial\bar{y}}\right)\right]d\bar{x}\,d\bar{y}\right\}\bar{c}_{kl}$$

$$+\iint\left\{\frac{\partial\bar{z}_p}{\partial\bar{x}}\left[3\frac{\partial\phi_k}{\partial\bar{x}}\frac{\partial\phi_m}{\partial\bar{x}}\frac{\partial\phi_i}{\partial\bar{x}}\psi_l\psi_n\psi_j + \left(\frac{a}{b}\right)^2\phi_k\phi_m\frac{\partial\phi_i}{\partial\bar{x}}\frac{\partial\psi_l}{\partial\bar{y}}\frac{\partial\psi_n}{\partial\bar{y}}\psi_j\right.\right.$$

$$+ 2\left(\frac{a}{b}\right)^2 \phi_k \frac{\partial \phi_m}{\partial \bar{x}} \phi_i \frac{\partial \psi_l}{\partial \bar{y}} \psi_n \frac{\partial \psi_j}{\partial \bar{y}}\Bigg]$$

$$+ \left(\frac{a}{b}\right)^2 \frac{\partial \bar{z}_p}{\partial \bar{y}} \Bigg[3\left(\frac{a}{b}\right)^2 \phi_k \phi_m \phi_i \frac{\partial \psi_l}{\partial \bar{y}} \frac{\partial \psi_n}{\partial \bar{y}} \frac{\partial \psi_j}{\partial \bar{y}} + \frac{\partial \phi_k}{\partial \bar{x}} \frac{\partial \phi_m}{\partial \bar{x}} \phi_i \psi_l \psi_n \frac{\partial \psi_j}{\partial \bar{y}}$$

$$+ 2 \frac{\partial \phi_k}{\partial \bar{x}} \phi_m \frac{\partial \phi_i}{\partial \bar{x}} \psi_l \frac{\partial \psi_n}{\partial \bar{y}} \psi_j \Bigg]\Bigg\} d\bar{x}\, d\bar{y}\, \bar{c}_{kl} \bar{c}_{mn}$$

$$+ \Bigg\{ \int \frac{\partial \phi_k}{\partial \bar{x}} \frac{\partial \phi_m}{\partial \bar{x}} \frac{\partial \phi_o}{\partial \bar{x}} \frac{\partial \phi_i}{\partial \bar{x}} d\bar{x} \int \psi_l \psi_n \psi_p \psi_j \, d\bar{y}$$

$$+ \left(\frac{a}{b}\right)^4 \int \phi_k \phi_m \phi_o \phi_i \, d\bar{x} \int \frac{\partial \psi_l}{\partial \bar{y}} \frac{\partial \psi_n}{\partial \bar{y}} \frac{\partial \psi_p}{\partial \bar{y}} \frac{\partial \psi_j}{\partial \bar{y}} d\bar{y}$$

$$+ \left(\frac{a}{b}\right)^2 \int \phi_k \frac{\partial \phi_m}{\partial \bar{x}} \frac{\partial \phi_o}{\partial \bar{x}} \phi_i \, d\bar{x} \int \frac{\partial \psi_l}{\partial \bar{y}} \psi_n \psi_p \frac{\partial \psi_j}{\partial \bar{y}} d\bar{y}$$

$$+ \left(\frac{a}{b}\right)^2 \int \frac{\partial \phi_k}{\partial \bar{x}} \phi_m \phi_o \frac{\partial \phi_i}{\partial \bar{x}} d\bar{x} \int \psi_l \frac{\partial \psi_n}{\partial \bar{y}} \frac{\partial \psi_p}{\partial \bar{y}} \psi_j \, d\bar{y} \Bigg\} \bar{c}_{kl} \bar{c}_{mn} \bar{c}_{op}$$

$$+ 2\frac{a}{h} \Bigg\{ \Bigg[\int \frac{\partial \alpha_k}{\partial \bar{x}} \frac{\partial \phi_m}{\partial \bar{x}} \frac{\partial \phi_i}{\partial \bar{x}} d\bar{x} \int \beta_l \psi_n \psi_j \, d\bar{y} + \nu \left(\frac{a}{b}\right)^2 \int \frac{\partial \alpha_k}{\partial \bar{x}} \phi_m \phi_i \, d\bar{x} \int \beta_l \frac{\partial \psi_n}{\partial \bar{y}} \frac{\partial \psi_j}{\partial \bar{y}} d\bar{y}$$

$$+ \frac{(1-\nu)}{2} \left(\frac{a}{b}\right)^2 \int \alpha_k \phi_m \frac{\partial \phi_i}{\partial \bar{x}} d\bar{x} \int \frac{\partial \beta_l}{\partial \bar{y}} \frac{\partial \psi_n}{\partial \bar{y}} \psi_j \, d\bar{y}$$

$$+ \frac{(1-\nu)}{2} \left(\frac{a}{b}\right)^2 \int \alpha_k \frac{\partial \phi_m}{\partial \bar{x}} \phi_i \, d\bar{x} \int \frac{\partial \beta_l}{\partial \bar{y}} \psi_n \frac{\partial \psi_j}{\partial \bar{y}} d\bar{y} \Bigg] \bar{c}_{mn}$$

$$+ \iint \Bigg[\frac{\partial \bar{z}_p}{\partial \bar{x}} \left(\frac{\partial \alpha_k}{\partial \bar{x}} \frac{\partial \phi_i}{\partial \bar{x}} \beta_l \psi_j + \frac{(1-\nu)}{2} \left(\frac{a}{b}\right)^2 \alpha_k \phi_i \frac{\partial \beta_l}{\partial \bar{y}} \frac{\partial \psi_j}{\partial \bar{y}} \right)$$

$$+ \left(\frac{a}{b}\right)^2 \frac{\partial \bar{z}_p}{\partial \bar{y}} \left(\nu \frac{\partial \alpha_k}{\partial \bar{x}} \phi_i \beta_l \frac{\partial \psi_j}{\partial \bar{y}} + \frac{(1-\nu)}{2} \alpha_k \frac{\partial \phi_i}{\partial \bar{x}} \frac{\partial \beta_l}{\partial \bar{y}} \psi_j \right) \Bigg] d\bar{x}\, d\bar{y} \Bigg\} \bar{a}_{kl}$$

$$+ 2\frac{a}{b}\frac{a}{h} \Bigg\{ \Bigg[\left(\frac{a}{b}\right)^2 \int \alpha_k \phi_m \phi_i \, d\bar{x} \int \frac{\partial \beta_l}{\partial \bar{y}} \frac{\partial \psi_n}{\partial \bar{y}} \frac{\partial \psi_j}{\partial \bar{y}} d\bar{y}$$

$$+ \nu \int \alpha_k \frac{\partial \phi_m}{\partial \bar{x}} \frac{\partial \phi_i}{\partial \bar{x}} d\bar{x} \int \frac{\partial \beta_l}{\partial \bar{y}} \psi_n \psi_j \, d\bar{y}$$

$$+ \frac{(1-\nu)}{2} \int \frac{\partial \alpha_k}{\partial \bar{x}} \phi_m \frac{\partial \phi_i}{\partial \bar{x}} d\bar{x} \int \beta_l \frac{\partial \psi_n}{\partial \bar{y}} \psi_j \, d\bar{y}$$

$$+ \frac{(1-\nu)}{2} \int \frac{\partial \alpha_k}{\partial \bar{x}} \frac{\partial \phi_m}{\partial \bar{x}} \phi_i \, d\bar{x} \int \beta_l \psi_n \frac{\partial \psi_j}{\partial \bar{y}} d\bar{y} \Bigg] \bar{c}_{mn}$$

$$+ \iint \Bigg[\frac{\partial \bar{z}_p}{\partial \bar{x}} \left(\nu\, \alpha_k \frac{\partial \phi_i}{\partial \bar{x}} \frac{\partial \beta_l}{\partial \bar{y}} \psi_j + \frac{(1-\nu)}{2} \frac{\partial \alpha_k}{\partial \bar{x}} \phi_i \beta_l \frac{\partial \psi_j}{\partial \bar{y}} \right)$$

$$+ \frac{\partial \bar{z}_p}{\partial \bar{y}} \left(\left(\frac{a}{b} \right)^2 \alpha_k \phi_i \frac{\partial \beta_l}{\partial \bar{y}} \frac{\partial \psi_j}{\partial \bar{y}} + \frac{(1-\nu)}{2} \frac{\partial \alpha_k}{\partial \bar{x}} \frac{\partial \phi_i}{\partial \bar{x}} \beta_l \psi_j \right) \Bigg] d\bar{x}\, d\bar{y} \Bigg\} \bar{b}_{kl}$$

$$- \alpha (1+\nu) \left(\frac{a}{h} \right)^2 \Bigg\{ \iint \Delta T(x,y) \left[\frac{\partial \phi_k}{\partial \bar{x}} \frac{\partial \phi_i}{\partial \bar{x}} \psi_l \psi_j + \left(\frac{a}{b} \right)^2 \phi_k \phi_i \frac{\partial \psi_l}{\partial \bar{y}} \frac{\partial \psi_j}{\partial \bar{y}} \right] d\bar{x}\, d\bar{y}\, \bar{c}_{kl}$$

$$+ \iint \Delta T(x,y) \left[\frac{\partial \bar{z}_p}{\partial \bar{x}} \frac{\partial \phi_i}{\partial \bar{x}} \psi_j + \left(\frac{a}{b} \right)^2 \frac{\partial \bar{z}_p}{\partial \bar{y}} \phi_i \frac{\partial \psi_j}{\partial \bar{y}} \right] d\bar{x}\, d\bar{y} \Bigg\}$$

$$+ \frac{(1-\nu^2)}{Eh} \left(\frac{a}{h} \right)^2 \left\{ \left[\int \frac{\partial \phi_k}{\partial \bar{x}} \frac{\partial \phi_i}{\partial \bar{x}} d\bar{x} \int \psi_l \psi_j\, d\bar{y} \right] \bar{c}_{kl} + \iint \frac{\partial \bar{z}_p}{\partial \bar{x}} \frac{\partial \phi_i}{\partial \bar{x}} \psi_j\, d\bar{x}\, d\bar{y} \right\} \left(N_x^a \right)$$

$$+ \frac{(1-\nu^2)}{Eh} \left(\frac{a}{b} \right)^2 \left(\frac{a}{h} \right)^2 \left\{ \left[\int \phi_k \phi_i\, d\bar{x} \int \frac{\partial \psi_l}{\partial \bar{y}} \frac{\partial \psi_j}{\partial \bar{y}} d\bar{y} \right] \bar{c}_{kl} + \iint \frac{\partial \bar{z}_p}{\partial \bar{y}} \phi_i \frac{\partial \psi_j}{\partial \bar{y}} d\bar{x}\, d\bar{y} \right\} \left(N_y^a \right)$$

$$+ \frac{1}{6D} \frac{a}{b} \left(\frac{a}{h} \right)^2 \bar{Q}_{zij} \qquad 0 \tag{43}$$

SECTION V

Nonconservative Forces

Q_{zij} are the nonconservative applied transverse loads. In this analysis, the loading is due to a static pressure differential between the plate's upper and lower surfaces, the aerodynamic flow over the upper surface of the plate, and structural damping. From linear theory, Q_{zij} can be written as

$$Q_{zij} = \iint \left\{ \Delta P_S + (P - P_\infty) + P_{Damp} \right\} \frac{\partial w}{\partial c_{ij}} \, dx \, dy \tag{44}$$

where ΔP_S is the static pressure differential, $(P - P_\infty)$ is the supersonic aerodynamic pressure differential, and P_{Damp} is a structural damping load. The static pressure differential is positive when the upper surface of the plate is in tension.

The supersonic aerodynamic pressure differential is written as

$$P - P_\infty = \frac{\rho_\infty U_\infty^2}{\beta} \left\{ \frac{\partial w}{\partial x} + \left[\frac{M^2 - 2}{M^2 - 1} \right] \frac{1}{U_\infty} \frac{\partial w}{\partial t} \right\} + \frac{\rho_\infty U_\infty^2}{\beta} \frac{\partial z_p}{\partial x} \tag{45}$$

where the term within the braces comes from first-order piston theory and the last term is a static pressure due to the plate's initial out-of-plane shape. Equation (45) is derived in Section VI.

The mathematical model used to represent the structural damping is that of viscous damping

$$P_{Damp} = C_{VD} \frac{\partial w}{\partial t} \tag{46}$$

Substituting the nondimensional parameters into (44), (45), and (46), and combining the results yields

$$\bar{Q}_{zij} = hab \iint \left\{ \Delta P_S + \frac{\rho_\infty U_\infty^2}{\beta} \frac{h}{a} \left[\frac{\partial \bar{w}}{\partial \bar{x}} + \left(\frac{M^2 - 2}{M^2 - 1} \right) \frac{a}{U_\infty} \left(\frac{D_x}{m_p a^4} \right)^{1/2} \frac{\partial \bar{w}}{\partial \tau} \right] + \frac{\rho_\infty U_\infty^2}{\beta} \frac{h}{a} \frac{\partial \bar{z}_p}{\partial \bar{x}} \right.$$
$$\left. + \, C_{VD} h \left(\frac{D_x}{m_p a^4} \right)^{1/2} \frac{\partial \bar{w}}{\partial \tau} \right\} \frac{\partial \bar{w}}{\partial \bar{c}_{ij}} \, d\bar{x} \, d\bar{y}$$

and substituting in the nondimensional dynamic pressure and mass ratio expressions

$$\lambda = \frac{\rho_\infty U_\infty^2 a^3}{D_x} \qquad \mu = \frac{\rho_\infty a}{\rho_p h}$$

produces

$$\bar{Q}_{zij} = \frac{\lambda D_x}{\beta} \frac{b}{a} \left(\frac{h}{a} \right)^2 \iint \left\{ \frac{\beta a^3}{\lambda D_x} \frac{a}{h} \Delta P_S + \frac{\partial \bar{w}}{\partial \bar{x}} + \left(\frac{M^2 - 2}{M^2 - 1} \right) \left(\frac{\mu}{\lambda} \right)^{1/2} \frac{\partial \bar{w}}{\partial \tau} + \frac{\partial \bar{z}_p}{\partial \bar{x}} \right.$$

$$+ C_{VD}\left(\frac{\mu}{\lambda}\right)^{1/2}\frac{\beta}{\rho_\infty U_\infty}\frac{\partial\bar{w}}{\partial\tau}\bigg\}\frac{\partial\bar{w}}{\partial\bar{c}_{ij}}\,d\bar{x}\,d\bar{y}$$

The final step is to replace $\bar{w}$ with its nondimensional modal expansion, (26). The nonconservative forces on the plate written in terms of a specific ij are

$$\begin{aligned}
\bar{Q}_{zij} &= a^3\frac{b}{a}\frac{h}{a}\Delta P_S\int\phi_i\,d\bar{x}\int\psi_j\,d\bar{y} \\
&+ \frac{\lambda D_x}{\beta}\frac{b}{a}\left(\frac{h}{a}\right)^2\bigg\{\left[\int\frac{\partial\phi_k}{\partial\bar{x}}\phi_i\,d\bar{x}\int\psi_l\psi_j\,d\bar{y}\right]\bar{c}_{kl} \\
&\qquad + \left(\frac{M^2-2}{M^2-1}\right)\left(\frac{\mu}{\lambda}\right)^{1/2}\left[\int\phi_k\phi_i\,d\bar{x}\int\psi_l\psi_j\,d\bar{y}\right]\dot{\bar{c}}_{kl} + \int\!\!\int\frac{\partial\bar{z}_p}{\partial\bar{x}}\phi_i\psi_j\,d\bar{x}\,d\bar{y}\bigg\} \\
&+ C_{VD}\,a^2\left(\frac{D_x}{m_p}\right)^{1/2}\frac{b}{a}\left(\frac{h}{a}\right)^2\left[\int\phi_k\phi_i\,d\bar{x}\int\psi_l\psi_j\,d\bar{y}\right]\dot{\bar{c}}_{kl} \qquad (47)
\end{aligned}$$

SECTION VI

Unsteady Aerodynamics

Unsteady supersonic aerodynamic forces are the dynamic driving force in a panel flutter analysis. Many different theories could be used to describe these forces but in this analysis piston theory aerodynamics will be used. The nonlinear form is derived from the conservation equations. Several assumptions are made pertaining to the flow: (1) the isentropic relations and the perfect gas equation of state apply, (2) gravity effects are negligible, (3) the flow is irrotational, and (4) the fluid may be treated as a continuum. This derivation uses information gathered from Shapiro [Ref. 8 and 9].

The conservation of momentum equation (including the irrotational assumption) is

$$\frac{\partial \vec{V}}{\partial t} + \frac{1}{2}\nabla V^2 = -\frac{1}{\rho}\nabla P \tag{48}$$

Since the flow is irrotational, its velocity can be represented by a potential function as

$$\vec{V} = \nabla\Phi \tag{49}$$

Substituting the velocity potential into the conservation of momentum produces

$$\frac{\partial \nabla\Phi}{\partial t} + \frac{1}{2}\nabla(\nabla\Phi)^2 = -\frac{1}{\rho}\nabla P \tag{50}$$

After taking the dot product of (50) with a unit vector along a streamline, integrate from an upstream reference point where the flow is steady to some point over the panel

$$\int_{\infty} d\left(\frac{\partial\Phi}{\partial t}\right) + \int_{\infty} d\left[\frac{1}{2}(\nabla\Phi)^2\right] + \int_{\infty}\frac{1}{\rho}dP = 0$$

which yields

$$\frac{\partial\Phi}{\partial t} + \frac{1}{2}(\nabla\Phi)^2 + \int_{\infty}\frac{1}{\rho}dP = \frac{1}{2}U_{\infty}^2 \tag{51}$$

where U_{∞} is the uniform and steady flow velocity at the reference point.

Since the flow is isentropic, the isentropic pressure/density relation

$$\rho = \rho_{\infty}\left(\frac{P}{P_{\infty}}\right)^{1/\gamma} \tag{52}$$

can be substituted into the integral term of Equation (51)

$$\int_{\infty}\frac{1}{\rho}dP = \frac{P_{\infty}^{1/\gamma}}{\rho_{\infty}}\int_{\infty} P^{-1/\gamma}dP = \frac{\gamma}{\gamma-1}\frac{P_{\infty}}{\rho_{\infty}}\left[\left(\frac{P}{P_{\infty}}\right)^{(\gamma-1)/\gamma} - 1\right]$$

Substituting this and the speed of sound definition

$$c_\infty^2 = \frac{\gamma P_\infty}{\rho_\infty}$$

into Equation (51) yields

$$\frac{\partial \Phi}{\partial t} + \frac{1}{2}(\nabla\Phi)^2 + \frac{c_\infty^2}{\gamma - 1}\left[\left(\frac{P}{P_\infty}\right)^{(\gamma-1)/\gamma} - 1\right] = \frac{1}{2}U_\infty^2 \tag{53}$$

Also note that the isentropic pressure/speed of sound relation

$$\left(\frac{c}{c_\infty}\right)^2 = \left(\frac{P}{P_\infty}\right)^{(\gamma-1)/\gamma} \tag{54}$$

could be introduced at this time

$$\frac{\partial \Phi}{\partial t} + \frac{1}{2}(\nabla\Phi)^2 + \frac{c_\infty^2}{\gamma - 1}\left[\left(\frac{c}{c_\infty}\right)^2 - 1\right] = \frac{1}{2}U_\infty^2 \tag{55}$$

The flow ahead of a piston in a tube is assumed to be one-dimensional. Piston theory aerodynamics assumes that a vibrating structure in a supersonic flow is analogous to an infinite number of oscillating infinitesimal pistons placed side by side forming the shape of the structure. It further assumes that the flow changes due to one piston do not effect the flow over an adjacent piston. Thus, the perturbations to the flow over a point on a panel will be defined using one-dimensional theory. Equation (55) written in one spacial dimension is

$$\frac{\partial \Phi}{\partial t} + \frac{1}{2}\left(\frac{\partial \Phi}{\partial z}\right)^2 + \frac{c_\infty^2}{\gamma - 1}\left[\left(\frac{c}{c_\infty}\right)^2 - 1\right] = \frac{1}{2}U_\infty^2 \tag{56}$$

where it was defined earlier that z is the vertical direction. Solving Equation (56) for the speed of sound in the disturbed flow gives

$$c^2 = c_\infty^2 - (\gamma - 1)\left[\frac{\partial \Phi}{\partial t} + \frac{1}{2}\left(\frac{\partial \Phi}{\partial z}\right)^2 - \frac{1}{2}U_\infty^2\right] \tag{57}$$

Since the flow over the panel is one-dimensional,

$$\dot{w}(x,y,t) = \frac{\partial \Phi}{\partial z} \tag{58}$$

where w is the vertical displacement of the point on the panel being evaluated.

The one-dimensional conservation of mass equation written in terms of the velocity potential is

$$\frac{\partial \rho}{\partial t} + \frac{\partial \Phi}{\partial z}\frac{\partial \rho}{\partial z} + \rho\frac{\partial^2 \Phi}{\partial z^2} = 0 \tag{59}$$

Differentials of the one-dimensional form of Equation (51) will be used to eliminate density from the conservation of mass equation. Differentiating Equation (51) with respect to z produces

$$\frac{\partial^2\Phi}{\partial z\partial t} + \frac{\partial\Phi}{\partial z}\frac{\partial^2\Phi}{\partial z^2} + \frac{1}{\rho}\frac{dP}{dz} = 0$$

and solving for the pressure derivative yields

$$\frac{dP}{dz} = \frac{dP}{d\rho}\frac{\partial\rho}{\partial z} = -\rho\left[\frac{\partial^2\Phi}{\partial z\partial t} + \frac{\partial\Phi}{\partial z}\frac{\partial^2\Phi}{\partial z^2}\right] \tag{60}$$

The isentropic pressure/density equation, (52), can be rewritten in differential form as

$$\frac{dP}{d\rho} = \frac{\gamma P_\infty}{\rho_\infty^\gamma}\rho^{\gamma-1} \tag{61}$$

which is then substituted into (60) to get

$$\frac{\partial\rho}{\partial z} = -\frac{\rho^2}{\gamma P_\infty}\left(\frac{\rho_\infty}{\rho}\right)^\gamma\left[\frac{\partial^2\Phi}{\partial z\partial t} + \frac{\partial\Phi}{\partial z}\frac{\partial^2\Phi}{\partial z^2}\right] \tag{62}$$

Differentiating the one-dimensional form of Equation (51) with respect to t leaves

$$\frac{\partial^2\Phi}{\partial t^2} + \frac{\partial\Phi}{\partial z}\frac{\partial^2\Phi}{\partial z\partial t} + \frac{1}{\rho}\frac{dP}{dt} = 0$$

and solving this expression for the pressure derivative

$$\frac{dP}{dt} = \frac{dP}{d\rho}\frac{\partial\rho}{\partial t} = -\rho\left[\frac{\partial^2\Phi}{\partial t^2} + \frac{\partial\Phi}{\partial z}\frac{\partial^2\Phi}{\partial z\partial t}\right] \tag{63}$$

Equation (63) can be rewritten using (61)

$$\frac{\partial\rho}{\partial t} = -\frac{\rho^2}{\gamma P_\infty}\left(\frac{\rho_\infty}{\rho}\right)^\gamma\left[\frac{\partial^2\Phi}{\partial t^2} + \frac{\partial\Phi}{\partial z}\frac{\partial^2\Phi}{\partial z\partial t}\right] \tag{64}$$

Substituting (62) and (64) into the conservation of mass equation, (59), yields

$$\left[\left(\frac{\gamma P_\infty}{\rho_\infty^\gamma}\right)\rho^{\gamma-1} - \left(\frac{\partial\Phi}{\partial z}\right)^2\right]\frac{\partial^2\Phi}{\partial z^2} - 2\frac{\partial\Phi}{\partial z}\frac{\partial^2\Phi}{\partial z\partial t} - \frac{\partial^2\Phi}{\partial t^2} = 0 \tag{65}$$

Using the isentropic relations, (52) and (54), Equation (65) is rewritten as

$$\left[c^2 - \left(\frac{\partial\Phi}{\partial z}\right)^2\right]\frac{\partial^2\Phi}{\partial z^2} - 2\frac{\partial\Phi}{\partial z}\frac{\partial^2\Phi}{\partial z\partial t} - \frac{\partial^2\Phi}{\partial t^2} = 0 \tag{66}$$

Equation (66) is a nonlinear, homogeneous partial differential equation of second order which describes the unsteady motion of the flow over an oscillating piston.

The theory of characteristics can be used to solve partial differential equations (PDE) that are hyperbolic and have the form

$$A\Phi_{zz} + 2B\Phi_{zt} + C\Phi_{tt} = 0 \tag{67}$$

where

$$\Phi_{zz} = \frac{\partial^2\Phi}{\partial z^2} \qquad \Phi_{zt} = \frac{\partial^2\Phi}{\partial z \partial t} \qquad \Phi_{tt} = \frac{\partial^2\Phi}{\partial t^2}$$

and the coefficients A, B, and C are functions of z, t, Φ_z, and Φ_t . Equation (66) is in this form and is hyperbolic since $B^2 - AC = c^2 > 0$. The hyperbolic nature of the PDE dictates that there are two real characteristics that satisfy Equation (66). A characteristic describes the motion of a pressure wave through a fluid.

The flow over the panel, therefore, consists of simple waves since its PDE has the form shown in (67). These simple waves are pressure waves emanating from the oscillating piston. Their amplitude may be small or large.

The equation describing the motion of each wave must satisfy the governing PDE. There are two families of waves that satisfy the PDE. There are upward- and downward-traveling pressure waves (traveling with respect to a fluid particle). The nomenclature assumes the piston is oscillating in a vertical, constant cross-sectional area duct oriented such that the upward direction is positive. Either one of these pressure wave families may exist in a flow, but not both at one time.

A flow that contains simple waves is defined such that Φ_z and Φ_t are unique functions of each other [Ref. 9]. Thus, the derivatives in Equation (67) can be rewritten as

$$\Phi_{zz} = \frac{\partial \Phi_z}{\partial z} = \frac{d\Phi_z}{d\Phi_t}\frac{\partial \Phi_t}{\partial z}$$

$$\Phi_{zt} = \frac{\partial \Phi_z}{\partial t} = \frac{\partial \Phi_t}{\partial z}$$

$$\Phi_{tt} = \frac{\partial \Phi_t}{\partial t} = \frac{d\Phi_t}{d\Phi_z}\frac{\partial \Phi_z}{\partial t} = \frac{d\Phi_t}{d\Phi_z}\frac{\partial \Phi_t}{\partial z}$$

and substituting these into (67) gives

$$A + 2B\frac{d\Phi_t}{d\Phi_z} + C\left(\frac{d\Phi_t}{d\Phi_z}\right)^2 = 0$$

Solving for $\dfrac{d\Phi_t}{d\Phi_z}$ results in

$$\frac{d\Phi_t}{d\Phi_z} = \frac{-B \pm \sqrt{B^2 - AC}}{C}$$

and after substituting in the values of the coefficients from Equation (66)

$$\frac{d\Phi_t}{d\Phi_z} = -\left(\frac{\partial \Phi}{\partial z} \pm c\right) = -(\dot{w} \pm c) \qquad (68)$$

where the minus sign corresponds to upward-traveling waves and the plus sign to downward-traveling waves. Now, differentiating Equation (57) with respect to $\dot{w}$ yields

$$\frac{\partial c}{\partial \dot{w}} = -\frac{\gamma - 1}{2c}\left[\frac{d\Phi_t}{d\dot{w}} + \dot{w}\right] = -\frac{\gamma - 1}{2c}\left[\frac{d\Phi_t}{d\Phi_z} + \dot{w}\right] \tag{69}$$

and substituting (68) into (69) produces

$$\frac{\partial c}{\partial \dot{w}} = \pm\frac{\gamma - 1}{2} \tag{70}$$

Integrating (70) from the upstream reference point to some point over the panel gives

$$c - c_\infty = \pm\frac{\gamma - 1}{2}\left(\dot{w} - \dot{w}_\infty\right)$$

Dividing through by c_∞ and noting that $\dot{w}_\infty = 0$ because the flow is uniform in the x direction at the reference point yields

$$\frac{c}{c_\infty} = 1 \pm \frac{\gamma - 1}{2}\frac{\dot{w}}{c_\infty} \tag{71}$$

Recall the isentropic relation given in Equation (54). Replacing the speed of sound ratio with the pressure ratio provides

$$\frac{P}{P_\infty} = \left(1 + \frac{\gamma - 1}{2}\frac{\dot{w}}{c_\infty}\right)^{\frac{2\gamma}{\gamma - 1}} \tag{72}$$

Equation (72) is written for upward-traveling waves only since both upward- and downward-traveling waves cannot exist at the same time. Equation (72) is the equation upon which piston theory is based [Ref. 1 and 7]. It relates the pressure on the face of a piston to its motion regardless of the magnitude of that motion.

Equation (72) can be expanded using a binomial series expansion to get different piston theory orders. First order piston theory is generally written as

$$P - P_\infty = \rho_\infty c_\infty \dot{w} \tag{73}$$

The second order binomial expansion gives

$$P - P_\infty = \rho_\infty c_\infty^2\left[\frac{\dot{w}}{c_\infty} + \frac{\gamma + 1}{4}\left(\frac{\dot{w}}{c_\infty}\right)^2\right] \tag{74}$$

And third order piston theory is

$$P - P_\infty = \rho_\infty c_\infty^2\left[\frac{\dot{w}}{c_\infty} + \frac{\gamma + 1}{4}\left(\frac{\dot{w}}{c_\infty}\right)^2 + \frac{\gamma + 1}{12}\left(\frac{\dot{w}}{c_\infty}\right)^3\right] \tag{75}$$

The expression defining the velocity of the piston surface, (58), needs to be discussed further. As stated in the Symbols section, $z_p(x, y)$ defines the initial shape of the plate and w is the vertical displacement measured relative to this initial shape. Since the plate is being divided into infinitesimal pistons, this function also describes the initial orientation of the piston surface. The vertical velocity of the piston in a uniform horizontal stream is then described by

$$\dot{w} = \frac{D(w + z_p)}{Dt} = \frac{\partial w}{\partial t} + U_\infty \frac{\partial w}{\partial x} + U_\infty \frac{\partial z_p}{\partial x} \tag{76}$$

Substituting the vertical displacement expression into the first order piston theory expression yields

$$P - P_\infty = \frac{\rho_\infty U_\infty^2}{M_\infty}\left(\frac{1}{U_\infty}\frac{\partial w}{\partial t} + \frac{\partial w}{\partial x} + \frac{\partial z_p}{\partial x}\right) \tag{77}$$

First order piston theory is used in this analysis since higher order aerodynamics do not affect the panel response unless the deflections are on the order of the plate length [Ref. 5] which is outside the limits of the structural assumptions imposed here.

Equation (77) can be employed when the Mach number is sufficiently high, $M > 2$, and the length-to-width ratio is low, $a/b < \beta$ [Ref. 3]. Equation (77) can be modified to include Mach numbers down to $\sqrt{2}$.

$$P - P_\infty = \frac{\rho_\infty U_\infty^2}{\beta}\left(\frac{M^2-2}{M^2-1}\frac{1}{U_\infty}\frac{\partial w}{\partial t} + \frac{\partial w}{\partial x} + \frac{\partial z_p}{\partial x}\right) \tag{78}$$

This form was derived from a first order expansion in reduced frequency of the exact two-dimensional, unsteady, linearized, potential flow equation. Equation (78) can be used to define the unsteady aerodynamic loads imposed on a structure due to a flow whose Mach number is greater than $\sqrt{2}$.

SECTION VII

Conclusions

The nondimensional equations of motion were derived for a thin panel using nonlinear strain-displacement and linear stress-strain expressions. The panel was assumed to have a constant thickness and a flat or slightly curved surface. Both orthotropic and isotropic materials were examined. Solving these equations would produce the in- and out-of-plane motion of the panel subject to external loads. The out-of-plane displacements were limited such that the local curvature anywhere on the panel would not be greater than 0.12 radians per panel length. Also, the magnitude of the out-of-plane displacements was assumed to be much larger than the magnitude of the in-plane displacements.

The boundary conditions on the panel were not defined other than a fluid flow only over the upper surface. Thus, these equations are applicable to rectangular panels whose edges are any combination of the following boundary conditions; clamped, simply supported, and/or free. The boundary conditions are introduced through the nondimensional modal functions, Equation (26).

This derivation assumed that the panel was subject to static and dynamic external loads. Several different loading conditions were examined. The static loads consisted of in-plane forces, a temperature differential, and a pressure differential. An unsteady supersonic flow over the upper surface of the panel was modelled using first order piston theory. Structural damping was incorporated using a viscous damping model which only affected the out-of-plane motion. The equations were written such that other aerodynamic and damping models could be readily substituted for the current ones.

The equations describing the motion of an orthotropic panel subject to the above in-plane loads and some arbitrary out-of-plane loads were defined in Equations (31), (32), and (33). Their derivation and the assumptions made during the derivation were discussed.

Equations (41), (42), and (43) described the isotropic panel motion in the x, y, and z directions, respectively, and Equation (47) defined some possible nonconservative transverse forces acting on the panel in the z direction.

A large amplitude simple pressure wave expression, Equation (72), was derived from the conservation equatons using the method of characteristics. This expression described the pressure on the surface of a piston oscillating in a one dimensional duct. This was the basis from which first order piston theory aerodynamics, Equation (73), was derived.

This analysis did not consider panels with large initial curvature (shells) or with an underlying cavity. Nor did this derivation discuss the ply arrangement for a laminated material with orthotropic properties. These parameters and other aerodynamic theories will be included in future research.

SECTION VIII

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SECTION IX

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